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Derived Equivalences of Generalized Kummer Varieties

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Introduction

In this thesis we are concerned with derived equivalences of certain hyperkähler varieties, namely generalized Kummer varieties. For simplicity, we work in this introduction over an algebraically closed field $\Bbbk$ of characteristic zero.

The subject of algebraic geometry studies the geometry of algebraic varieties. We want to pose the vague dichotomy that distinguishes their *inner* and *outer* geometry. Here the inner geometry looks at how a variety is assembled from pieces and at objects which are constructed *inside* it, like subvarieties or cycles. In contrast, the outer geometry is concerned with objects defined/parametrized *on* the variety, like functions, vector bundles, or coherent sheaves.

Part of the goal of algebraic geometry is the classification of varieties. But since a precise classification of isomorphism classes of varieties is vastly too ambitious, one is lead to consider notions of equivalence that are weaker than isomorphisms. From the inner point of view, two varieties X and Y are *birationally equivalent* if two Zariski-dense open subvarieties $U \subset X$ and $V \subset Y$ are isomorphic. From the outer point of view, one collects all coherent sheaves on X in the derived category

$$\mathbf{D}^{\mathrm{b}}(X) \coloneqq \mathbf{D}^{\mathrm{b}}(\mathbf{Coh}(X)),$$

which one can view as a *noncommutative* incarnation of X, cf. [KS09; Kal09], and asks when $\mathbf{D}^{\mathrm{b}}(X)$ and $\mathbf{D}^{\mathrm{b}}(Y)$ are equivalent triangulated categories. Recall that the derived category of coherent sheaves arises from the category of complexes of coherent sheaves by inverting quasi-isomorphisms, cf. [HuyFM, Ch. 2]. It carries the structure of a triangulated category and was originally introduced by Verdier as a technical device to do homological algebra and work with derived functors, cf. [Ver77; Ver96]. One can consider as well the (unbounded) derived category $\mathbf{D}(A)$ of modules on an associative algebra or a dg-algebra A, and in this noncommutative sense the unbounded derived category $\mathbf{D}(X)$ is in fact always affine, i.e. $\mathbf{D}(X) \simeq \mathbf{D}(A)$, cf. [BB03, Cor. 3.1.8].

Derived equivalences. — Derived categories of coherent sheaves and their equivalences are intricate objects which gain their importance for example from their role in *homological mirror symmetry*, their rich interaction with *moduli spaces of stable sheaves*, and since they are nowadays considered as invariants of the variety in their own right.

Let X and Y be smooth projective varieties over $\Bbbk$. One says that X and Y are *derived equivalent* if $\mathbf{D}^{\mathrm{b}}(X)$ and $\mathbf{D}^{\mathrm{b}}(Y)$ are equivalent as $\Bbbk$-linear triangulated categories, that is there exists an equivalence

$$\Phi\colon\ \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$$

which is $\Bbbk$-linear and preserves distinguished triangles as well as shifts. One has to be precise with this definition, otherwise one runs the risk to obtain a notion that is as rigid as the notion of being isomorphic. For example if the derived equivalence Φ furthermore preserves the monoidal structure given by the derived tensor product on X and Y respectively, then X and Y are already isomorphic, cf. [Bal02]. Let us mention that X can also be reconstructed from the abelian category of coherent sheaves $\mathbf{Coh}(X)$ by Gabriel's theorem [Gab62], so it would be too restrictive to require that Φ preserves the natural t-structures. By Orlov [Orl97], a derived equivalence $\Phi\colon\ \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ can be written as a *Fourier–Mukai functor*

$$\mathrm{FM}_{\mathcal{P}}(-) \coloneqq \mathbf{R}\mathrm{pr}_{Y,*}(\mathbf{L}\mathrm{pr}_X^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{P})$$

associated to a unique Fourier–Mukai kernel $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(X \times Y)$, where $\mathrm{pr}_X\colon X \times Y \to X$ and $\mathrm{pr}_Y\colon X \times Y \to Y$ denote the coordinate projections. So derived equivalences are more geometric in nature than their abstract definition would suggest at first sight.

Nevertheless, by a result of Bondal–Orlov [BO01], if two varieties X and Y with ample or anti-ample canonical sheaf ω_X are derived equivalent, then they are already isomorphic. So it is natural to investigate the contrasting case of varieties X with trivial canonical sheaf $\omega_X \simeq \mathcal{O}_X$. By the Beauville–Bogomolov decomposition theorem [Bog74; Bea83], there are three building blocks for these varieties:

1) abelian varieties,
2) strict[(1)] Calabi–Yau varieties,
3) hyperkähler varieties.

That is, X can be decomposed, up to a finite étale cover, as a product of varieties of these kinds. We refer to [HN11] for results and discussion about the interaction between this decomposition and derived equivalences.

The first derived equivalence of non-isomorphic (even non-birational) varieties was provided by Mukai [Muk81], who explained that an abelian variety A and its dual $A^\vee$ are derived equivalent, while usually not isomorphic; the kernel is provided by the Poincaré bundle $\mathcal{P}$ on $A \times A^\vee$. More generally, derived equivalences of abelian varieties are well-understood by work of Mukai, Polishchuk, and Orlov [Muk81; Pol96; Orl02]. They explain equivalences between two abelian varieties A and B in terms of *symplectic isomorphisms*, which are isomorphisms $A \times A^\vee \to B \times B^\vee$ of abelian varieties satisfying certain properties, cf. Definition 2.3.2. We denote Orlov's set, respectively group, of symplectic isomorphisms by $\mathrm{Sp}'(A,B)$, or $\mathrm{Sp}'(A)$ when $A = B$, cf. Notation 2.3.9. Let us also denote the group of isomorphism classes of derived autoequivalences

(1) A *Calabi–Yau* variety X is a smooth, proper variety with $\omega_X \simeq \mathcal{O}_X$. A *strict* Calabi–Yau variety satisfies moreover $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $1 \leq i \leq \dim(X) - 1$.

of A by $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A))$, and let $\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A), \mathbf{D}^{\mathrm{b}}(B))$ denote the $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A))$-torsor of isomorphism classes of derived equivalences between A and B.

***Theorem* (Orlov [Orl02, §§0–4, Thm. 4.14]).** — *We have a short exact sequence of groups*

$$0 \to \mathbb{Z} \times A \times A^{\vee} \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A)) \to \mathrm{Sp}'(A) \to 0,$$

and a compatible, in the sense of natural torsor actions, surjective map

$$\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A), \mathbf{D}^{\mathrm{b}}(B)) \twoheadrightarrow \mathrm{Sp}'(A, B).$$

In contrast, not much is known about derived equivalences of strict Calabi–Yau varieties. By [Bri02] two strict Calabi–Yau threefolds which are birationally equivalent are also derived equivalent. This establishes a special case of the following conjecture of Bondal–Orlov [BO95] and Kawamata [Kaw02; Kaw18].

***Conjecture* (DK-hypothesis).** — *If X and Y are K-equivalent, i.e. there exists a smooth projective variety Z together with birational morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ such that $f^*\omega_X \simeq g^*\omega_Y$, then X and Y are derived equivalent.*

This conjecture does certainly not explain every derived equivalence of varieties, say with trivial canonical bundle, as the case of abelian varieties already shows (recall that birationally equivalent abelian varieties are automatically isomorphic). We will come back to this in the situation of generalized Kummer varieties below.

Derived equivalences of hyperkähler varieties. — The hyperkähler case remains tractable yet interesting. Just like elliptic curves or K3 surfaces, hyperkähler varieties enjoy a rich geometry which at the same time is amenable to concrete studies, e.g. via lattice theory, and thus provides an approachable testing ground for algebraic geometry.

Hyperkähler varieties in dimension 2 are nothing other than K3 surfaces, and much has been done in this case, see [HuyK3, Ch. 16] for an overview. We want to discuss Kummer K3 surfaces in a bit of detail, since on the one hand they are a precursor to generalized Kummer varieties and on the other hand we want to contrast the results about them with the higher dimensional case later on. Given an abelian surface A, the associated Kummer K3 surface $\mathrm{Kum}^1(A)$ is the minimal resolution of singularities of the quotient $A/\langle -1 \rangle$ of A by the negation involution.

***Theorem* (Hosono–Lian–Oguiso–Yau [HLOY03, Thm. 0.1]).** — *Let A and B be abelian surfaces. Then we have*

$$\mathbf{D}^{\mathrm{b}}(A) \simeq \mathbf{D}^{\mathrm{b}}(B) \tag{0.0.1}$$

if and only if

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^1(A)) \simeq \mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^1(B)). \tag{0.0.2}$$

In fact, this is also equivalent to

$$\mathrm{Kum}^1(A) \simeq \mathrm{Kum}^1(B). \tag{0.0.3}$$

The second logical equivalence "(0.0.2) $\iff$ (0.0.3)" follows from the fact that one is dealing with K3 surfaces of Picard rank $\rho(\mathrm{Kum}^1(A)) = \rho(A)+16 > 11$; such K3 surfaces are isomorphic as soon as their transcendental lattices are Hodge isometric, cf. [Muk87, Prop. 6.2]. The first logical equivalence "(0.0.1) $\iff$ (0.0.2)" is established using the derived Torelli theorems for K3 surfaces and abelian surfaces, and by comparing their transcendental lattices, cf. [BM01]. Unfortunately, this approach cannot be carried over to the case of generalized Kummer varieties discussed below. Stellari offers the following theorem, which generalizes the case of Kummer surfaces, where one can write $\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^1(A)) \simeq \mathbf{D}^{\mathrm{b}}([A/\langle -1\rangle])$ as the derived category of coherent sheaves on the quotient stack $[A/\langle -1\rangle]$; see the discussion about the equivariant approach below for more on this viewpoint.

***Theorem* (Stellari [Ste07, Thm. 1.1]).** — *If A and B are abelian varieties (not necessarily surfaces) which are derived equivalent, i.e. $\mathbf{D}^{\mathrm{b}}(A) \simeq \mathbf{D}^{\mathrm{b}}(B)$, then we still have an equivalence of derived categories*

$$\mathbf{D}^{\mathrm{b}}([A/\langle -1\rangle]) \simeq \mathbf{D}^{\mathrm{b}}([B/\langle -1\rangle])$$

of coherent sheaves on the respective quotient stacks.

In higher dimensions, there are four known types of hyperkähler varieties up to deformations, namely Hilbert schemes of points $\mathrm{Hilb}^n(S)$ where S is a K3 surface, generalized Kummer varieties $\mathrm{Kum}^m(A)$ where A is an abelian surface, and O'Grady's sporadic examples of dimension 6 and of dimension 10, cf. [Bea83; OGr99; OGr03]. Let us briefly recall the definition of generalized Kummer varieties $\mathrm{Kum}^{n-1}(A)$, where A is an abelian surface. Beauville [Bea83] defines them as a fiber of the morphism

$$\Sigma \circ \mathrm{HC}\colon\ \mathrm{Hilb}^n(A) \to A,$$

where $\mathrm{HC}\colon\ \mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A)$ is the Hilbert–Chow morphism and $\Sigma\colon \mathrm{Sym}^n(A) \to A$ denotes the summation map. The morphism $\Sigma \circ \mathrm{HC}$ is in fact an Albanese map of $\mathrm{Hilb}^n(A)$, cf. [Fog68, §3], and it is an isotrivial fibration. Beauville verifies then that the fibers are hyperkähler varieties.

Regarding derived equivalences of Hilbert schemes of points, Ploog has the following result.

***Theorem* (Ploog [Plo07, §3.1, Prop. 8]).** — *Let S and S' be two smooth, projective surfaces. Then there exists an injective group homomorphism* $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(S)) \hookrightarrow \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n(S)))$, *and any derived equivalence $\mathbf{D}^{\mathrm{b}}(S) \simeq \mathbf{D}^{\mathrm{b}}(S')$ induces (via the inflation map* (0.0.4) *below) a derived equivalence*

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n(S)) \simeq \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n(S')).$$

In particular, since an abelian surface A and its dual abelian surface $A^\vee$ are derived equivalent, we have $\mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n(A)) \simeq \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n(A^\vee))$. We want to attribute the following question to Namikawa [Nam02a; Nam02b], who studied the birational Torelli problem for hyperkähler varieties and remarked on their derived equivalences in this context.

***Question* (Namikawa).** — Are the generalized Kummer varieties $\mathrm{Kum}^m(A)$ and $\mathrm{Kum}^m(A^\vee)$ derived equivalent?

This question appears natural in view of Ploog's result about Hilbert schemes of points and the evidence about Kummer surfaces, where we have seen that $\mathrm{Kum}^1(A)$ and $\mathrm{Kum}^1(A^\vee)$ are in fact isomorphic. In this thesis we concern ourselves with this question and reach the following answer as our first main theorem.

***Theorem 1* (Theorems 6.1.8 and 6.2.1).** — *Let A be an abelian surface, let $m \in \mathbb{N}$ be even and assume that there exists a polarization $\lambda\colon A \to A^\vee$ whose exponent $\mathrm{e}(\lambda)$ satisfies*

$$\gcd(m+1, \mathrm{e}(\lambda)) = 1.$$

Then there exist a derived equivalence

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^m(A)) \simeq \mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^m(A^\vee)).$$

When $\mathrm{End}(A) = \mathbb{Z}$, the assumption reduces to $\gcd(m+1, \deg(\lambda)) = 1$, where $\lambda\colon A \to A^\vee$ is the polarization of minimal degree. In this situation the theorem is sharp in the sense that $\mathrm{Kum}^m(A)$ and $\mathrm{Kum}^m(A^\vee)$ cannot be derived equivalent via any “inflated” equivalence unless $\gcd(m+1, \deg(\lambda)) = 1$ is satisfied, cf. Remark 7.

Namikawa [Nam02a] showed that in general two generalized Kummer fourfolds $\mathrm{Kum}^2(A)$ and $\mathrm{Kum}^2(A^\vee)$ are not birationally equivalent. Following Okawa's [Oka21] study of the non-birationality of Hilbert schemes of points on K3 surfaces, we treat the case of generalized Kummer varieties and obtain in particular new examples of non-K-equivalent but derived equivalent varieties, i.e. counterexamples to the converse of the DK-hypothesis. For more hyperkähler examples of this kind (i.e. derived equivalent but not K-equivalent) see [ADM16; MMY20].

***Theorem 2* (Theorem 7.2.8).** — *There exist generalized Kummer varieties which are derived equivalent but which are not birationally equivalent.*

More precisely, let A be an abelian surface with $\mathrm{End}(A) = \mathbb{Z}$, and let $\deg(\lambda) = d^2$ be the minimal degree of a polarization $\lambda\colon A \to A^\vee$. If $\gcd(3, d) = 1$, and $4x^2 - 3dy^2 = 1$ has an integer solution(2)*, then the Kummer fourfolds $\mathrm{Kum}^2(A)$ and $\mathrm{Kum}^2(A^\vee)$ are not birationally equivalent.*

Equivariant approach. — We want to explain the approach underlying Ploog's and Stellari's results above as well as our Theorem 1.

The following alternative description of generalized Kummer varieties is more suitable for our investigations. Consider the kernel $A \otimes \Gamma_n$ of the summation map $\Sigma\colon A^{\times n} \to A$, where Γ_n is the kernel of the summation map $\mathbb{Z}^n \to \mathbb{Z}$, cf. §4.1. (The reader who desires so can understand $A \otimes \Gamma_n$ as an instance of Serre's tensor constructions, cf. [Con04, §7], [Ami18, §1].) The symmetric group S_n acts by coordinate

(2) A more general condition which is sufficient for the theorem is that some solution (x_0, y_0) of the Pell equation $x^2 - 3dy^2 = 1$ satisfies that y_0 is odd.

permutations on $A^{\times n}$ and trivially on A, so S_n acts on $A \otimes \Gamma_n$ (and similarly on Γ_n), and we get a diagram of fiber sequences

$$\begin{tikzcd} A \otimes \Gamma_n \arrow[r, hook] \arrow[d] & A^{\times n} \arrow[r, two heads, "\Sigma"] \arrow[d] & A \arrow[d] \\ (A \otimes \Gamma_n)/\mathrm{S}_n \arrow[r, hook] & \mathrm{Sym}^n(A) \arrow[r, two heads, "\Sigma"] & A. \end{tikzcd}$$

Now $\mathrm{Sym}^n(A)$ is singular, when $n \geq 2$, but it admits a crepant resolution in form of the Hilbert–Chow morphism $\mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A)$. Haiman [Hai01] provides an identification

$$\mathrm{Hilb}^n(A) \simeq \mathrm{Hilb}^{\mathrm{S}_n}(A^{\times n})$$

of the classical Hilbert scheme of points on A with Nakamura's equivariant Hilbert scheme of clusters on $A^{\times n}$, cf. [IN96; Rei97]. Similarly $(A \otimes \Gamma_n)/\mathrm{S}_n$ is singular, but it admits a crepant resolution of singularities

$$\mathrm{Hilb}^{\mathrm{S}_n}(A \otimes \Gamma_n) \to (A \otimes \Gamma_n)/\mathrm{S}_n,$$

which is isomorphic to the generalized Kummer variety $\mathrm{Kum}^{n-1}(A)$ associated to A.

To study the derived category of $\mathrm{Kum}^{n-1}(A)$, the derived McKay correspondence [BKR01] comes into play, which in this case says that we have equivalences of derived categories

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n(A)) \simeq \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^{\mathrm{S}_n}(A^{\times n})) \simeq \mathbf{D}^{\mathrm{b}}_{\mathrm{S}_n}(A^{\times n})$$

and

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^{n-1}(A)) \simeq \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^{\mathrm{S}_n}(A \otimes \Gamma_n)) \simeq \mathbf{D}^{\mathrm{b}}_{\mathrm{S}_n}(A \otimes \Gamma_n)$$

respectively. Here $\mathbf{D}^{\mathrm{b}}_{\mathrm{S}_n}(-)$ denotes the derived category of S_n-equivariant coherent sheaves $(\mathcal{F}, \lambda)$, which consists of complexes of coherent sheaves $\mathcal{F}$ endowed with an S_n-equivariant structure λ (see below on page xvi for a definition); it can also be viewed as the derived category $\mathbf{D}^{\mathrm{b}}([-/\mathrm{S}_n])$ of coherent sheaves on the quotient stack $[-/\mathrm{S}_n]$.

Remark. — At this point the reader might wonder why the derived equivalence of $A \otimes \Gamma_n$ and $(A \otimes \Gamma_n)^\vee$ (induced by the Poincaré bundle) does not immediately yield a derived equivalence of generalized Kummer varieties $\mathrm{Kum}^{n-1}(A)$ and $\mathrm{Kum}^{n-1}(A^\vee)$. Indeed, while $(A \otimes \Gamma_n)^\vee$ is isomorphic as an abelian variety to $A^\vee \otimes \Gamma_n$, the S_n-actions are different. Instead, we have $(A \otimes \Gamma_n)^\vee \simeq A^\vee \otimes \Gamma_n^\vee$, and $\Gamma_n \not\simeq \Gamma_n^\vee$ are non-isomorphic $\mathbb{Z}[\mathrm{S}_n]$-modules for $n \geq 3$. See [Nam02b] and [Plo05, §4.4] for related discussions.

Let G be a finite group acting on two smooth projective varieties X and Y. Ploog's method [Plo05; Plo07] to construct derived equivalences of equivariant derived categories consists roughly of the following three steps:

1) Exhibit G-invariant derived equivalences.
2) Endow the kernels of these G-invariant equivalences with an G-equivariant structure.
3) "Inflate" these G-equivariant equivalences to equivalences of equivariant derived categories.

Let us explain more precisely what these steps mean. We consider the following three sets of derived equivalences: First, we have the set of *G-invariant* derived equivalences between $\mathbf{D}^{\mathrm{b}}(X)$ and $\mathbf{D}^{\mathrm{b}}(Y)$, which are the equivalences commuting with the G-action up to isomorphism. In terms of Fourier–Mukai kernels, this is

$$\begin{aligned}&\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))^G\\ &\quad\simeq \{\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(X \times Y) \mid \mathrm{FM}_{\mathcal{P}}\colon \mathbf{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(Y), \text{ and } \forall g \in G\colon (g,g)^*\mathcal{P} \simeq \mathcal{P}\}/\simeq.\end{aligned}$$

Second, we have the set of derived equivalences between $\mathbf{D}^{\mathrm{b}}_G(X)$ and $\mathbf{D}^{\mathrm{b}}_G(Y)$. These are represented by kernels which are endowed with an equivariant structure $\widetilde{\lambda}$ for the $(G \times G)$-action on $X \times Y$, so

$$\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}_G(X), \mathbf{D}^{\mathrm{b}}_G(Y)) \simeq \{(\widetilde{\mathcal{P}}, \widetilde{\lambda}) \in \mathbf{D}^{\mathrm{b}}_{G\times G}(X \times Y) \mid \mathrm{FM}_{(\widetilde{\mathcal{P}},\widetilde{\lambda})}\colon \mathbf{D}^{\mathrm{b}}_G(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_G(Y)\}/\simeq.$$

Third, interpolating between the two cases above, we have the set of derived equivalences $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(Y)$ which are endowed with an equivariant structure λ witnessing that Φ "commutes coherently" with the G-action. Again in terms of kernels, this is

$$\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))^{\mathrm{h}G} := \{(\mathcal{P}, \lambda) \in \mathbf{D}^{\mathrm{b}}_{\Delta G}(X \times Y) \mid \mathrm{FM}_{\mathcal{P}}\colon \mathbf{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(Y)\}/\simeq,$$

where $\Delta G \subset G \times G$ denotes the diagonal subgroup. The notation "hG" is inspired by the concept of homotopy fixed points.

***Theorem* (Ploog [Plo07, Thm. 6]).** — *The obstruction to endow a G-invariant kernel of an equivalence with a G-equivariant structure takes values in the Schur multiplier, i.e. in the sequence*

$$\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))^{\mathrm{h}G} \xrightarrow{\mathrm{for}} \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))^G \xrightarrow{\delta_{X,Y}} \mathrm{H}^2(G, \Bbbk^\times)$$

the image of the forgetful map for*, which forgets the equivariant structure, equals the kernel of the obstruction map $\delta_{X,Y}$.*

Furthermore, there exists an inflation *map*

$$\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))^{\mathrm{h}G} \xrightarrow{\mathrm{inf}_{\Delta G}^{G\times G}} \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}_G(X), \mathbf{D}^{\mathrm{b}}_G(Y)) \tag{0.0.4}$$

associated to the subgroup $\Delta G \subset G \times G$, which maps a derived equivalence endowed with a G-equivariant structure to an equivalence of G-equivariant derived categories.

The upshot is that in the context of generalized Kummer varieties and Namikawa's question, we want to exhibit S_n-invariant derived equivalences between $A \otimes \Gamma_n$ and $A^\vee \otimes \Gamma_n$, and study their obstruction to admit an equivariant structure. We will attack this problem in the discussion surrounding Theorem 6.

Autoequivalences of generalized Kummer varieties. — We also study autoequivalences of generalized Kummer varieties in the style of Orlov's short exact sequence

$$0 \to \mathbb{Z} \times A \times A^\vee \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A)) \to \mathrm{Sp}'(A) \to 0$$

for abelian varieties, and Ploog's short exact sequence for Kummer surfaces, which we recall next.

***Theorem* (Ploog [Plo07, §3.2]).** — *Let A be an abelian surface, and recall that its subgroup of 2-torsion points is denoted by $A[2]$. Then we have a short exact sequence*

$$0 \to \mathbb{Z} \times A[2] \times A^\vee[2] \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A))^{\langle -1\rangle} \to \mathrm{Sp}'(A) \to 0,$$

and the group of invariant autoequivalences fits into the diagram

$$\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A))^{\langle -1\rangle} \twoheadleftarrow \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A))^{\mathrm{h}\langle -1\rangle} \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^1(A))),$$

where both maps have kernels isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

The short exact sequence in the theorem arises by applying group cohomology to Orlov's sequence, where the action is induced from the negation involution of the abelian variety A. The second diagram in the theorem is then an application of Ploog's method, taking into consideration the equivariant viewpoint on generalized Kummer varieties. For generalized Kummer varieties we can prove the following.

***Theorem 3* (Theorem 5.2.4).** — *Assume that n is odd, then we have a short exact sequence*

$$0 \to \mathbb{Z} \times A[n] \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n))^{\mathrm{S}_n} \to \mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n} \to 0.$$

If $n \neq 2, 4$ is even, we have an exact sequence of pointed sets

$$0 \to \mathbb{Z} \times A[n] \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n))^{\mathrm{S}_n} \to \mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n} \xrightarrow{\delta} A[2].$$

For $n = 4$, we have an exact sequence of pointed sets

$$0 \to \mathbb{Z} \times A[4] \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_4))^{\mathrm{S}_4} \to \mathrm{Sp}'(A \otimes \Gamma_4)^{\mathrm{S}_4} \xrightarrow{\delta} A[2] \times A^\vee[2].$$

We invite the reader to contrast these sequences with Ploog's sequence for Kummer surfaces above. In general, δ might not be a homomorphism, but in the case $\mathrm{End}(A) = \mathbb{Z}$ we remark that one can study this issue more precisely, see Theorem 5.2.4 for details. Furthermore, we calculate groups of S_n-invariant symplectic isomorphisms as follows.

***Theorem 4* (Propositions 5.1.10 and 5.1.13).** —

(i) *For every polarization $\lambda\colon A \to A^\vee$ of exponent $e = \mathrm{e}(\lambda)$ we have an inclusion*

$$\Gamma_0(ne) \subset \mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}, \tag{0.0.5}$$

where $\Gamma_0(ne) \subset \mathrm{SL}(2,\mathbb{Z})$ denotes the Hecke congruence subgroup of level ne.

(ii) *If $\gcd(n,e) = 1$, we have*

$$\mathrm{Sp}'(A \otimes \Gamma_n, A^\vee \otimes \Gamma_n)^{\mathrm{S}_n} \neq \emptyset$$

and it becomes a right-torsor under $\mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}$.

(iii) *If $\mathrm{End}(A) = \mathbb{Z}$, we can consider a polarization λ_0 of minimal degree $d = \mathrm{e}(\lambda_0)^2$. Then the inclusion (0.0.5) becomes an equality, and the condition $\gcd(n,d) = 1$ in (ii) becomes necessary in addition to being sufficient.*

The sequences in Theorem 3 are the result of applying non-abelian group cohomology to Orlov's short exact sequence with A replaced by $A \otimes \Gamma_n$. So the computation of the first group cohomology $\mathrm{H}^1(\mathrm{S}_n, \mathbb{Z} \times (A \otimes \Gamma_n) \times (A \otimes \Gamma_n)^\vee)$ is a key ingredient.

One consequence of Theorems 3 and 4 is that they provide many derived autoequivalences of generalized Kummer varieties and thus they serve as a lower bound for determining the full group of autoequivalences $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^m(A)))$ or its image under the cohomology representation

$$\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^m(A))) \to \mathrm{GL}(\mathrm{H}^\bullet(\mathrm{Kum}^m(A)), \mathbb{Q}).$$

Recently Taelman [Tae19] and Beckmann [Bec23] determined the image of the cohomology representation up to finite index for certain hyperkähler varieties of $\mathrm{K3}^{[n]}$-deformation type. Crucial for the lower bound inclusion in their computation is a large supply of derived equivalences coming from derived equivalences of K3 surfaces via the derived McKay correspondence.

At this point, let us mention in passing the work of Meachan [Mea15] and Krug–Meachan [KM17] on autoequivalences of generalized Kummer varieties induced by $\mathbb{P}$-functors, which act trivially on cohomology, cf. [Add16, §3.4].

Integral representation theory of symmetric groups. — We undertake a systematic study of the standard representation Γ_n of the symmetric group S_n, as well as of its dual $\Gamma_n^\vee$, with the particular goal of computing their first group cohomology. This falls into the realm of integral/modular representation theory. Fortunately, the representation theory of the symmetric group is well studied and its group cohomology with trivial coefficients was computed in the 1960's by topologists, who where motivated by the consequences for the Steenrod algebra. One aspect of these computations, which plays a particular role in our study, is Nakaoka's cohomological stability theorem.

***Theorem* (Nakaoka [Nak60, Thm. 5.8, Cor. 6.7]).** — *Let A be an abelian group endowed with trivial S_n-action. Then the restriction map*

$$\mathrm{res}^{\mathrm{S}_n}_{\mathrm{S}_{n-1}} \colon \mathrm{H}^k(\mathrm{S}_n, A) \to \mathrm{H}^k(\mathrm{S}_{n-1}, A)$$

is an isomorphism for $k < n/2$; it is always surjective.

Nowadays this theorem has been revisited and re-proven several times [Qui74; RW17; SW20; Kup21] and belongs to the area of *homological stability* [Wah22], which provides powerful tools for computing the (co)homology of certain families of groups.

Regarding coefficients with non-trivial action, one can find in the literature computations of some Ext-groups in the category of $\Bbbk[\mathrm{S}_n]$-modules, but with the caveat that $\Bbbk$ is assumed to be a field, e.g. [KS99; Shc04; Hem09; CHN10]. Recall that group cohomology can be viewed as such Ext-groups, namely $\mathrm{H}^\bullet(G, -) \simeq \mathrm{Ext}^\bullet_{\Bbbk G}(\Bbbk, -)$. We calculate the group cohomology $\mathrm{H}^\bullet(\mathrm{S}_n, \Gamma_n \otimes_{\mathbb{Z}} A)$ in Nakoaka's stable range with arbitrary coefficients in an abelian group A in terms of the group cohomology $\mathrm{H}^\bullet(\mathrm{S}_n, A)$ of the symmetric group.

***Theorem 5* (Propositions 4.2.5 and 4.2.7).** — *Let A be an abelian group endowed with trivial S_n-action. For $k < n/2$ we have short exact sequences of group cohomology groups*

$$0 \to \mathrm{H}^{k-1}(\mathrm{S}_n, A)/n\,\mathrm{H}^{k-1}(\mathrm{S}_n, A) \to \mathrm{H}^k(\mathrm{S}_n, \Gamma_n \otimes A) \to \mathrm{H}^k(\mathrm{S}_n, A^{\oplus n})[n] \to 0$$

and

$$0 \to \mathrm{H}^k(\mathrm{S}_n, \Gamma_n^\vee \otimes A) \to \mathrm{H}^{k+1}(\mathrm{S}_n, A) \xrightarrow{\mathrm{res}} \mathrm{H}^{k+1}(\mathrm{S}_{n-1}, A) \to 0.$$

Furthermore, for $k < n/2 - 1$, we have the identity

$$\mathrm{H}^k(\mathrm{S}_n, \Gamma_n^\vee \otimes A) = 0.$$

This allows us to compute the desired first group cohomology groups, where A is taken to be the group of rational points of an abelian variety. It turns out that group cohomology is not only useful in the study of sequences involving *groups* of autoequivalences, but by taking the viewpoint that the non-abelian group cohomology group $\mathrm{H}^1(G, \mathit{\Gamma})$ classifies G-equivariant $\mathit{\Gamma}$-torsors, see §3.2, one can also study fixed points of *torsors* of derived equivalences. Using this we arrive at the following theorem.

***Theorem 6* (Theorem 6.1.5). —** *Assume that n is odd, and let $\lambda\colon A \to A^\vee$ be some polarization of exponent* $\mathrm{e}(\lambda)$*. If* $\gcd(n, \mathrm{e}(\lambda)) = 1$*, then*

$$\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n), \mathbf{D}^{\mathrm{b}}(A^\vee \otimes \Gamma_n))^{\mathrm{S}_n} \neq \emptyset \tag{0.0.6}$$

and it is a right-torsor under $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n))^{\mathrm{S}_n}$*. If* $\mathrm{End}(A) = \mathbb{Z}$*, the converse "*(0.0.6) *implies* $\gcd(n, \mathrm{e}(\lambda)) = 1$*" is true when we take λ to be the polarization of minimal degree.*

Equivariant structures and linearization obstructions. — Keeping Ploog's method in mind, to deduce Theorem 1, it is left to study when an invariant Fourier–Mukai kernel admits an equivariant structure.

Let us recall a bit more precisely the definition of an equivariant structure. Let G be a finite group acting on two smooth projective varieties X and Y over the field $\Bbbk$. A G-equivariant object $(\mathcal{F}, \lambda) \in \mathbf{D}^{\mathrm{b}}_G(X)$ is an object $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ endowed with a G-equivariant structure λ (also called a *G-linearization*, especially in the case of line bundles), which is given by isomorphisms $\lambda_g\colon \mathcal{F} \xrightarrow{\sim} g^*\mathcal{F}$ for each $g \in G$, subject to the cocycle condition that $\lambda_1 = \mathrm{id}_{\mathcal{F}}$ and $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$ for $g, h \in G$. We say that $\mathcal{F}$ is G-invariant, if the isomorphisms λ_g do not necessarily satisfy the cocycle condition. Similar to the obstruction for a projective representation to be linearized, the obstruction for a G-invariant object $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ which satisfies $\mathrm{End}(\mathcal{F}) = \Bbbk$ to admit a G-equivariant structure lies in the Schur multiplier $\mathrm{H}^2(G, \Bbbk^\times)$. Thus, following Ploog, we have a *linearization obstruction* homomorphism

$$\delta_X\colon \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))^G \to \mathrm{H}^2(G, \Bbbk^\times)$$

for autoequivalences, and a similar map

$$\delta_{X,Y}\colon \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))^G \to \mathrm{H}^2(G, \Bbbk^\times),$$

which are compatible with each other, taking the natural $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))^G$-torsor action into account.

For symmetric groups, we have $\mathrm{H}^2(\mathrm{S}_n, \Bbbk^\times) = 0$ for $n \leq 3$ and $\mathrm{H}^2(\mathrm{S}_n, \Bbbk^\times) \simeq \mathbb{Z}/2\mathbb{Z}$ for $n \geq 4$. So for generalized Kummer varieties of dimension 2 and 4 there is no linearization obstruction to consider at all. Up to this point we could blackbox

Orlov's construction underlying his short exact sequence, but to get control over the linearization obstructions (for $n \geq 4$) we need extra information and thus we work in §6.2 step by step through Orlov's proof and Mukai's construction [Muk78] of semi-homogeneous vector bundles on which Orlov relies.

***Question* (Ploog [Plo05, Qst. 3.21]).** — Is the obstruction map $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))^G \to \mathrm{H}^2(G, \Bbbk^\times)$ surjective in general?

A positive answer to this question would have been helpful, since then any S_n-invariant autoequivalence in $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))^{\mathrm{S}_n}$ with non-trivial linearization obstruction could be used, by composing with it, to kill any non-trivial obstruction of an equivalence in $\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))^{\mathrm{S}_n}$. But, we claim to know that one can answer Ploog's question negatively. More precisely, we claim that the obstruction homomorphism

$$\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n))^{\mathrm{S}_n} \to \mathrm{H}^2(\mathrm{S}_n, \Bbbk^\times)$$

is the zero map if A is an abelian variety satisfying $\mathrm{End}(A) = \mathbb{Z}$. The proof is omitted from this thesis for reasons of space and exposition.

Concluding remarks and questions. —

Remark 7. — Finally, we want to remark that Theorem 1 is sharp in the following sense: Let G be a finite group acting on two varieties X and Y. A derived equivalence $\mathbf{D}^{\mathrm{b}}_G(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_G(Y)$ is called *inflated* if its Fourier–Mukai kernel lies in the image of the inflation map $\mathrm{inf}_{\Delta G}^{G\times G}$ from Ploog's theorem on page xiii. Assume $n \geq 3$ and $\mathrm{End}(A) = \mathbb{Z}$, and let d denote the minimal degree of a polarization of an abelian surface A. Then $\mathrm{Kum}^{n-1}(A)$ and $\mathrm{Kum}^{n-1}(A^\vee)$ cannot be derived equivalent via an inflated equivalence unless $\gcd(n, d) = 1$. This is because the relevant set of invariant symplectic isomorphisms in Theorem 4(ii) needs to be non-empty, which by Theorem 4(iii) is equivalent to $\gcd(n, d) = 1$.

Question. — Is it necessary for $\mathrm{Kum}^m(A)$ and $\mathrm{Kum}^m(A^\vee)$ to be derived equivalent that there exists an isogeny $\lambda\colon A \to A^\vee$ of exponent coprime to $m+1$?

There is some weak evidence towards this question, or rather an arithmetic analog of it: Frei–Honigs [FH23, Cor. 1.2] announced an example of a generalized Kummer fourfold $\mathrm{Kum}^2(A)$ over a number field K which cannot be derived equivalent to $\mathrm{Kum}^2(A^\vee)$ over the field K, and the abelian variety A cannot carry a polarization defined over K whose degree is coprime to 3, since $A[3] \not\simeq A^\vee[3]$.

Outline. — This thesis is divided into three parts. The first part recalls some general theory, but the results are set up in a way to make the arguments in Part II very efficient.

- In particular we review in Chapter 1 the algebraic geometry of generalized Kummer varieties and abelian varieties, as well as semi-homogeneous vector bundles on the latter.

- Next we recall in Chapter 2 derived categories and their Fourier–Mukai equivalences, also in an equivariant setting, and consider the particular cases of abelian varieties and Kummer surfaces.
- Then we discuss in Chapter 3 the general theory of group cohomology with coefficients in an abelian group, as well as some results in non-abelian group cohomology in connection with equivariant torsors.

This concludes the first part of the thesis and the second part is concerned with the proofs of our main theorems.

- In Chapter 4 we study in detail the standard representation of the symmetric group and calculate its group cohomology and arrive at Theorem 5.
- Using this, we deduce Theorem 3 in Chapter 5, which is about S_n-invariant derived equivalences, and we compute the S_n-fixed points in the group/set of symplectic auto-/iso-morphisms of an abelian variety, yielding Theorem 4.
- Finally we consider in Chapter 6 derived equivalences of dual generalized Kummer varieties by first exhibiting S_n-invariant equivalences, proving Theorem 6, and afterwards working through Orlov's and Mukai's construction to get control over the linearization obstruction, thus arriving at Theorem 1.

In Part III we collect miscellaneous results and prove in particular:

- Theorem 2 about non-birationality of dual generalized Kummer varieties.
- A description of the automorphism group of certain generalized Kummer varieties.

Conventions. — We use the following conventions throughout this thesis.

- Fields are denoted by $\Bbbk$. We make no global assumptions about their characteristic or algebraically closedness. Rather, such assumptions are made per section in "Situation" paragraphs.
- The assumptions and notations in a "Situation" paragraphs are in force till the end of the section in which they appear.
- A variety X over a field $\Bbbk$ is a scheme X which is separated and of finite type over $\mathrm{Spec}(\Bbbk)$.

Part I

Fundamentals and preliminaries

CHAPTER 1

Generalized Kummer varieties and abelian varieties

1.1. Hyperkähler varieties and generalized Kummer varieties

We review some facts about hyperkähler varieties. This section is expository, but we provide a proof regarding the crepantness of the Hilbert–Chow morphism of a generalized Kummer variety which aligns well with the calculational nature of later sections of this thesis. See [Huy99] for a much more comprehensive summary of classical results about hyperkähler varieties.

1.1.1. Situation. — We work over an algebraically closed field $\Bbbk$ of characteristic 0.

***1.1.2. Definition* (Hyperkähler varieties).** — A *hyperkähler variety* is a smooth projective variety X over $\Bbbk$ such that

(i) $\mathrm{H}^0(X, \Omega^2_{X/\Bbbk})$ is generated by a nowhere degenerate 2-form σ, and
(ii) X is simply connected, i.e. $\pi_1^{\text{ét}}(X, \mathrm{pt}) = 0$.

1.1.3. — A hyperkähler variety X is even dimensional, since the nowhere degenerate 2-form σ endows the tangent spaces of X with the structure of a symplectic vector space. The canonical sheaf ω_X of X is trivial since the Pfaffian of σ provides a nowhere vanishing global section of the line bundle ω_X.

1.1.4. Remark. — If $\Bbbk = \mathbb{C}$ is the field of complex numbers, one can replace the étale fundamental group in Definition 1.1.2 by the topological fundamental group $\pi_1(X^{\mathrm{an}}, \mathrm{pt})$. Indeed, if $\pi_1(X^{\mathrm{an}}, \mathrm{pt})$ is zero then also its profinite completion $\pi_1^{\text{ét}}(X, \mathrm{pt})$ is zero; following [Bin21, Lem. 3.1.3], the converse implication can be deduced from the Beauville–Bogomolov decomposition theorem, cf. [Bea83, Thm. 1].

***1.1.5. Example* (K3 surfaces).** — In dimension 2 a hyperkähler variety is nothing other than a *K3 surface*, which is defined as a smooth projective surface such that $\Omega^2_{X/\Bbbk} \simeq \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

Indeed, condition (i) in Definition 1.1.2 just means that the line bundle $\omega_X \simeq \Omega^2_{X/\Bbbk}$ has a nowhere vanishing section. Regarding (ii) we have by Hodge symmetry that $\dim_\Bbbk \mathrm{H}^1(X, \mathcal{O}_X) = \dim_\Bbbk \mathrm{H}^0(X, \Omega^1_{X/\Bbbk})$ and the latter vanishes when X is a hyperkähler variety, cf. [Bea83, Prop. 3.(ii)][(1)]; conversely a K3 surface X has Euler characteristic $\chi(X, \mathcal{O}_X) = 2$, so any finite connected étale cover of X must be trivial by looking at its Euler characteristic, cf. [HuyK3, Rem. 1.2.3].

***1.1.6. Example* (Kummer surfaces).** — A particular class of K3 surfaces, which are generalized in Example 1.1.10 to higher dimensions, are the *Kummer surfaces*. Let A be an abelian surface and denote the negation involution of A by $[-1]\colon A \to A$. The fixed points of this involution are exactly the sixteen 2-torsion points $A[2] \subset A$, cf. Example 1.2.13. The quotient $A/\langle[-1]\rangle$ has rational double point singularities, and its minimal resolution

$$S \to A/\langle[-1]\rangle$$

is a K3 surface which is called a *Kummer surface* and denoted by $\mathrm{Kum}^1(A)$, cf. [HuyK3, Ex. 1.1.3.(iii)].

One can obtain this minimal resolution by first blowing up the 2-torsion points of A and afterwards taking the quotient by the involution induced by the negation map. Then the divisor classes $[\overline{E_i}]$ of the quotients of the 16 exceptional divisors E_i are a linearly independent set in the Néron–Severi group $\mathrm{NS}(S)$. In fact we have the identity

$$\rho(S) = \rho(A) + 16$$

for the Picard rank $\rho(S) = \mathrm{rk}(\mathrm{NS}(S))$, cf. [HuyK3, §3.2.5].

1.1.7. — Recall that a resolution of singularities $f\colon \widetilde{X} \to X$ is called *crepant* if $f^*\omega_X \simeq \omega_{\widetilde{X}}$, where ω_X (respectively $\omega_{\widetilde{X}}$) denotes the canonical sheaf of X (respectively of $\widetilde{X}$).[(2)] Note that a crepant resolution of a normal surface is automatically minimal and in particular it is unique. To see this use the ramification formula [IshIS, Thm. 6.1.7] (applied to a morphism from a crepant to a minimal resolution) and the description of the Picard group of a blowup as the direct sum of the Picard group of the base plus the classes of exceptional divisors.

The following proposition should be well-known to the experts; we provide a proof as a didactic preparation for Proposition 1.1.15 below.

1.1.8. Proposition. — *The resolution of singularities $S \to A/\langle[-1]\rangle$ in Example 1.1.6 is crepant.*

Proof. — For notation let $G \coloneqq \mathbb{Z}/2\mathbb{Z}$ act via $[-1]$ on A with quotient $A/\langle[-1]\rangle$. Since we know that S is a K3 surface, we have $\omega_S \simeq \mathcal{O}_S$, so it suffices to show that $\omega_{A/G} \simeq \mathcal{O}_{A/G}$. Set $A^\circ \coloneqq A \setminus A[2]$, then G acts freely on A° and the quotient map

(1) Alternatively, one could use the Hurewicz theorem in étale cohomology [FuECT, Prop. 5.7.20] to conclude that $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}_\ell) = 0$. Over $\mathbb{C}$ one can then use the comparison of étale and singular cohomology, and the Hodge decomposition to arrive at the desired vanishing $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

(2) To make sense of the canonical sheaf ω_X, either assume that X is quasi-projective and define ω_X as the dualizing sheaf, cf. [IshIS, Def. 5.3.5], or assume that X is normal and define $\omega_X \coloneqq \jmath_*\omega_{X_{\mathrm{reg}}}$, where $\jmath\colon X_{\mathrm{reg}} \hookrightarrow X$ is the inclusion of the non-singular locus. In the latter case one can also view ω_X as the divisorial sheaf associated to the closure of a canonical Weil divisor on X_{reg}.

$f\colon A^\circ \to A^\circ/G$ is étale. Since $\omega_{-/\Bbbk}$ is a sheaf on the small étale site of A°/G, the sheaf condition for the covering f provides the equalizer sequence[(3)]

$$0 \to \mathrm{H}^0(A^\circ/G, \omega_{-/\Bbbk}) \to \mathrm{H}^0(A^\circ, \omega_{-/\Bbbk}) \rightrightarrows \mathrm{H}^0(A^\circ \times G, \omega_{-/\Bbbk}),$$

where the upper map sends a form ω to $(\omega)_{g\in G}$ and the lower map sends it to $(g^*\omega)_{g\in G}$. This means that

$$\mathrm{H}^0(A^\circ/G, \omega_{-/\Bbbk}) \simeq \mathrm{H}^0(A^\circ, \omega_{-/\Bbbk})^G.$$

The abelian variety A has trivial cotangent bundle $\Omega^1_A \simeq \mathcal{O}_A \mathrm{d}z_1 \oplus \mathcal{O}_A \mathrm{d}z_2$, where $\mathrm{d}z_1, \mathrm{d}z_2$ shall be considered as abstract symbols (not to be confused with the differential of a non-existent global 0-form). So the canonical bundle ω_A is generated by the global nowhere vanishing form $\mathrm{vol} \coloneqq \mathrm{d}z_1 \wedge \mathrm{d}z_2$. Since $[-1]^* \mathrm{vol} = \mathrm{vol}$ shows that vol is G-invariant, it descends to a global nowhere vanishing form $\overline{\mathrm{vol}}$ on A°/G, which witnesses that $\omega_{A^\circ/G} \simeq \mathcal{O}_{A^\circ/G}$. Finally, since A/G is normal as the categorical quotient of a normal variety, we have $\omega_{A/G} \simeq j_*\omega_{A^\circ/G}$, where $j\colon A^\circ/G \to A/G$ denotes the open immersion, cf. [IshIS, Cor. 5.3.9], and by reflexivity of $\mathcal{O}_{A/G}$ we have $\mathcal{O}_{A/G} \simeq j_*\mathcal{O}_{A^\circ/G}$, cf. [IshIS, Thm. 5.1.11]. This shows that

$$\omega_{A/G} \simeq j_*\omega_{A^\circ/G} \simeq j_*\mathcal{O}_{A^\circ/G} \simeq \mathcal{O}_{A/G}. \qquad \square$$

***1.1.9. Example* (Hilbert schemes of points).** — For a more detailed explanation see [FGAex, Ch. 7]. Let S be a smooth projective surface. The symmetric group S_n acts on the variety $S^{\times n}$ by permuting the factors. The n-th *symmetric product* is defined as the quotient

$$\mathrm{Sym}^n(S) \coloneqq S^{\times n}/\mathrm{S}_n$$

and is a parameter space for effective 0-cycles on S of degree n, which are written as formal sums of points. Related to this construction is the *Hilbert scheme of points* $\mathrm{Hilb}^n(S)$ which is a projective variety parametrizing closed subschemes Z of S of length $\dim_\Bbbk \mathrm{H}^0(Z, \mathcal{O}_Z) = n$.

The symmetric product will be singular, cf. [FGAex, Ex. 7.1.3.(2)], and in this regard the Hilbert scheme of points on the surface is better behaved since it is smooth and irreducible, cf. [Fog68], [FGAex, §7.2]. They are related by the Hilbert–Chow morphism

$$\mathrm{HC}\colon \mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S), \quad Z \mapsto \sum_{p\in S} \dim(\mathcal{O}_{Z,p})[p],$$

which is in fact a resolution of singularities, cf. [Fog68], [FGAex, §7.1, Thm. 7.3.4].

If S has trivial canonical bundle $\omega_S \simeq \mathcal{O}_S$, Beauville shows that $\mathrm{Hilb}^n(S)$ admits a closed 2-form $\sigma \in \mathrm{H}^0(\mathrm{Hilb}^n(S), \Omega^2_{\mathrm{Hilb}^n(S)})$ which is nowhere degenerate, cf. [Bea83, §6, Prop. 5]. In fact the arguments generalize to show that the Hilbert–Chow morphism is a *crepant* resolution of singularities whenever S is a smooth surface. When S is a K3 surface, then $\mathrm{Hilb}^n(S)$ is simply connected and $\dim \mathrm{H}^0(\mathrm{Hilb}^n(S), \Omega^2_{\mathrm{Hilb}^n(S)}) = \dim \mathrm{H}^0(S, \Omega^2_S) = 1$, so $\mathrm{Hilb}^n(S)$ is a hyperkähler variety of dimension $2n$, cf. [Bea83, Prop. 6, Thm. 3]. In fact, over the complex numbers and assuming $n \geq 2$, we have an

(3) This argument was inspired from [MO, Answer 226902].

isomorphism of integral Hodge structures

$$\mathrm{H}^2(\mathrm{Hilb}^n(S),\mathbb{Z}) \simeq \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta, \tag{1.1.1}$$

where $\delta \in \mathrm{H}^{1,1}(\mathrm{Hilb}^n(S))$ is 1/2 times the class of the 'exceptional divisor', cf. [Bea83, Prop. 6, Rem.].

***1.1.10. Example* (Generalized Kummer varieties).** — Following [Bea83, §7], let A be an abelian surface. Then $\mathrm{Hilb}^n(A)$ is not a hyperkähler variety since it is not simply connected. Instead, since A carries a group structure, we can consider the summation morphism $\Sigma\colon A^{\times n} \to A$. It is equivariant for the permutation action on $A^{\times n}$, so it descends to a morphism $\mathrm{Sym}^n(A) \to A$, and via composition with the Hilbert–Chow morphism it becomes a morphism

$$\Sigma\colon \mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A) \to A,$$

which is in fact the Albanese map, cf. [Fog68, §3], [Yos01, §4.3.1]. The *generalized Kummer variety* $\mathrm{Kum}^{n-1}(A)$ is defined as the fiber $\Sigma^{-1}(0) \subset \mathrm{Hilb}^n(A)$.

1.1.11. Proposition. — *Let A be an abelian surface. Then the generalized Kummer variety $\mathrm{Kum}^{n-1}(A)$ is a hyperkähler variety of dimension $2(n-1)$.*

Proof. — We provide a sketch; for details see [Bea83, Prop. 7, Prop. 8, Thm. 4]. The summation map Σ is equivariant with respect to the translation action of A on $\mathrm{Hilb}^n(A)$ and the action $(a,b) \mapsto na+b$ of A on itself. So Σ is smooth and isotrivial, i.e. we have a pullback diagram

$$\begin{tikzcd} A \times \mathrm{Kum}^{n-1}(A) \arrow[r, "\mathrm{act}"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Hilb}^n(A) \arrow[d, "\Sigma"] \\ A \arrow[r, "{[n]}"] & A. \end{tikzcd}$$

Next, $\mathrm{Kum}^{n-1}(A)$ is simply connected: Beauville [Bea83, Lem. 1] explains that $\pi_1(\mathrm{Hilb}^n(A)) \to \pi_1(\mathrm{Sym}^n(A))$ is an isomorphism and that the homomorphism $\varphi\colon \pi_1(A) \to \pi_1(\mathrm{Sym}^n(A))$ induced from the inclusion of a factor $A \hookrightarrow A^{\times n} \to \mathrm{Sym}^n(A)$ is in fact the abelianization map. But since $\pi_1(A) \simeq \mathbb{Z}^{2n}$ is already abelian, φ is an isomorphism, and since φ splits the map $\Sigma_*\colon \pi_1(\mathrm{Sym}^n(A)) \to \pi_1(A)$, the latter is an isomorphism as well. Considering the long exact homotopy sequence associated to the fibration $\mathrm{Kum}^{n-1}(A) \to \mathrm{Hilb}^n(A) \to A$ and using that $\pi_2(A) \simeq \pi_2(\mathbb{S}^1)^{2n} = 0$, we see that $\mathrm{Kum}^{n-1}(A)$ is simply-connected.

The desired nowhere degenerate 2-form σ' on $\mathrm{Kum}^{n-1}(A)$ arises as the restriction of the 2-form σ on $\mathrm{Hilb}^n(A)$ from Example 1.1.9. If $n=2$, we are already done, since σ' witnesses that $\omega_{\mathrm{Kum}^1(A)} \simeq \Omega^2_{\mathrm{Kum}^1(A)} \simeq \mathcal{O}_{\mathrm{Kum}^1(A)}$ is the trivial line bundle. To check that σ' generates the space of 2-forms when $n \geq 3$, Beauville establishes an isomorphism of Hodge structures

$$\mathrm{H}^2(\mathrm{Kum}^{n-1}(A),\mathbb{C}) \simeq \mathrm{H}^2(A,\mathbb{C}) \oplus \mathbb{C}\delta,$$

where $\delta \in \mathrm{H}^{1,1}(\mathrm{Kum}^{n-1}(A))$. So we have

$$\dim \mathrm{H}^0(\mathrm{Kum}^{n-1}(A), \Omega^2) = \dim \mathrm{H}^0(A, \Omega^2_A) = \dim \mathrm{H}^0(A, \mathcal{O}_A) = 1.$$

The isomorphism of Hodge structures is proven along the following lines. The restriction map $\mathrm{H}^2(\mathrm{Hilb}^n(A), \mathbb{C}) \to \mathrm{H}^2(\mathrm{Kum}^{n-1}(A), \mathbb{C})$ is surjective, and together with the vanishing of $\mathrm{H}^1(\mathrm{Kum}^{n-1}(A), \mathbb{C})$ by simply-connectedness, the Serre spectral sequence leads to a short exact sequence

$$0 \to \mathrm{H}^2(A, \mathbb{C}) \xrightarrow{\Sigma^*} \mathrm{H}^2(\mathrm{Hilb}^n(A), \mathbb{C}) \to \mathrm{H}^2(\mathrm{Kum}^{n-1}(A), \mathbb{C}) \to 0.$$

The argument is finished by describing $\mathrm{H}^2(\mathrm{Hilb}^n(A), \mathbb{C})$ similarly as in (1.1.1). □

1.1.12. — For the description of generalized Kummer varieties via equivariant Hilbert schemes in ¶1.1.13, we need to discuss Nakamura's equivariant Hilbert scheme of clusters, cf. [IN96; Rei97; Blu11]. Let G be a finite group which acts faithfully on a reduced quasi-projective variety X. The equivariant Hilbert scheme $G\text{-Hilb}(X)$ is a fine moduli space parametrizing "G-clusters", i.e. 0-dimensional subschemes $Z \hookrightarrow X$ which are G-invariant and whose coordinate ring $\mathrm{H}^0(Z, \mathcal{O}_Z)$ is isomorphic to the regular representation $\mathbb{k}[G]$ of G. In particular, any G-cluster has length $\dim_{\mathbb{k}} \mathrm{H}^0(Z, \mathcal{O}_Z) = \#G$ and is supported on a G-orbit, but the latter may be endowed with a plethora of unreduced structures. An example of a general G-cluster is a free cluster, consisting of $\#G$ distinct points (indeed, by faithfulness of the action, there exists an open subset of X on which G acts freely).

As in the case of Hilbert schemes of points, there is a projective Hilbert–Chow morphism

$$\mathrm{HC}\colon G\text{-Hilb}(X) \to X/G, \quad Z \mapsto (Z/G)^{\mathrm{red}}$$

assigning to a G-cluster the orbit it is supported on. It is not clear whether $G\text{-Hilb}(X)$ is irreducible, but there is a preferred component containing the free G-clusters, which we denote by $\mathrm{Hilb}^G(X)$. Then the restriction of the Hilbert–Chow morphism

$$\mathrm{HC}\colon\ \mathrm{Hilb}^G(X) \to X/G$$

is a birational surjective morphism which is an isomorphism over the locus of free orbits.

One can recover the Hilbert scheme of points on a surface as an instance of the equivariant Hilbert scheme. Let S be a smooth projective surface, then Haiman [Hai01] provides an identification

$$\mathrm{Hilb}^n(S) \simeq \mathrm{Hilb}^{\mathrm{S}_n}(S^{\times n})$$

of the usual Hilbert scheme of points on S with an S_n-equivariant Hilbert scheme of clusters.

1.1.13. — Similarly to the description of a Hilbert scheme of points as a crepant resolution of the symmetric product in Example 1.1.9, one can also construct generalized Kummer varieties as resolutions of singularities of certain singular quotients.

Let A be an abelian surface. Consider the kernel $A \otimes \Gamma_n$ of the summation map $\Sigma\colon A^{\times n} \to A$, where Γ_n is defined to be the kernel of the summation map $\mathbb{Z}^n \to \mathbb{Z}$, see §4.1 for details. The symmetric group S_n acts by coordinate permutations on $A^{\times n}$ and trivially on A, so we get a diagram of fiber sequences

$$\begin{tikzcd} A \otimes \Gamma_n \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A^{\times n} \arrow[r, two heads, "\Sigma"] \arrow[d] & A \arrow[d, equal] \\ (A \otimes \Gamma_n)/\mathrm{S}_n \arrow[r, hook] & \mathrm{Sym}^n(A) \arrow[r, two heads, "\Sigma"] & A, \end{tikzcd}$$

where one verifies that the fiber in the bottom row is indeed the quotient $(A \otimes \Gamma_n)/\mathrm{S}_n$ using the fact that the morphism in the middle is a universal categorical quotient in characteristic zero, cf. [MFG, Thm. 1.1]. We have already seen that $\mathrm{Sym}^n(A)$ is singular, but it admits a crepant resolution in form of the Hilbert–Chow morphism $\mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A)$. By ¶1.1.12 we can make the identification $\mathrm{Hilb}^n(A) \simeq \mathrm{Hilb}^{\mathrm{S}_n}(A^{\times n})$. Similarly $(A \otimes \Gamma_n)/\mathrm{S}_n$ is singular, but it admits a crepant resolution of singularities

$$\mathrm{Hilb}^{\mathrm{S}_n}(A \otimes \Gamma_n) \to (A \otimes \Gamma_n)/\mathrm{S}_n,$$

which is once again the generalized Kummer variety $\mathrm{Kum}^{n-1}(A) \simeq \mathrm{Hilb}^{\mathrm{S}_n}(A \otimes \Gamma_n)$, cf. Propositions 1.1.14 and 1.1.15.

The following propositions seem to appear implicitly in the literature, so we consider it worthwhile to spell them out and provide arguments.

1.1.14. Proposition. — *We have an isomorphism* $\mathrm{Kum}^{n-1}(A) \simeq \mathrm{Hilb}^{\mathrm{S}_n}(A \otimes \Gamma_n)$ *which is compatible with the "Hilbert–Chow" morphisms* $\mathrm{Kum}^{n-1}(A) \to (A \otimes \Gamma_n)/\mathrm{S}_n$ *and* $\mathrm{Hilb}^{\mathrm{S}_n}(A \otimes \Gamma_n) \to (A \otimes \Gamma_n)/\mathrm{S}_n$.

Proof. — Using Haiman's identification, cf. ¶1.1.12, we have a cartesian diagram

$$\begin{tikzcd} \mathrm{Kum}^{n-1}(A) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Hilb}^{\mathrm{S}_n}(A^{\times n}) \arrow[d, "\mathrm{HC}"] \\ (A \otimes \Gamma_n)/\mathrm{S}_n \arrow[r, hook] & \mathrm{Sym}^n(A). \end{tikzcd}$$

The closed immersion $A \otimes \Gamma_n \hookrightarrow A^{\times n}$ induces a closed immersion $\mathrm{Hilb}^{\mathrm{S}_n}(A \otimes \Gamma_n) \hookrightarrow \mathrm{Hilb}^{\mathrm{S}_n}(A^{\times n})$, cf. [Blu11, Thm. 5.1] and [SP, Tag 0B97], which factors through the fiber product $\mathrm{Kum}^{n-1}(A)$ as a closed immersion $\varphi\colon \mathrm{Hilb}^{\mathrm{S}_n}(A \otimes \Gamma_n) \hookrightarrow \mathrm{Kum}^{n-1}(A)$. Since generalized Kummer varieties are reduced, it suffices to show that φ is surjective in order for it to be an isomorphism. But the image of φ is closed, since φ is proper, and it contains the dense open subset of free orbits, so it exhausts all of $\mathrm{Kum}^{n-1}(A)$. □

1.1.15. Proposition. — *The resolution of singularities* $\mathrm{Kum}^{n-1}(A) \simeq \mathrm{Hilb}^{\mathrm{S}_n}(A \otimes \Gamma_n) \to (A \otimes \Gamma_n)/\mathrm{S}_n$ *is crepant.*

Proof. — As in the proof of Proposition 1.1.8, it suffices to show that the canonical sheaf $\omega_{(A \otimes \Gamma_n)/\mathrm{S}_n}$ is trivial, since we already know that $\omega_{\mathrm{Kum}^{n-1}(A)}$ is trivial. Indeed, since $\Sigma\colon \mathrm{Hilb}^n(A) \to A$ is an isotrivial fibration, the normal bundle of the subvariety $\mathrm{Kum}^{n-1}(A) \subset \mathrm{Hilb}^n(A)$ is trivial, so

$$\omega_{\mathrm{Kum}^{n-1}(A)} \simeq \omega_{\mathrm{Hilb}^n(A)}|_{\mathrm{Kum}^{n-1}(A)} \simeq \mathcal{O}_{\mathrm{Kum}^{n-1}(A)}$$

by adjunction. Using the same argument on the level of symmetric products, we could conclude after checking that $\omega_{\mathrm{Sym}^n(A)}$ is trivial.

We follow a slightly more verbose proof, foreshadowing techniques that play an essential role in Part II. Using that Ω^1_A is free with basis denoted by the abstract symbols[(4)] $\mathrm{d}z$, $\mathrm{d}z'$, and considering n copies $\mathrm{d}z_i$, $\mathrm{d}z'_i$ of this basis, we see that $\Omega^1_{A^{\times n}}$ is free with basis $\mathrm{d}z_1, \mathrm{d}z'_1, \dots, \mathrm{d}z_n, \mathrm{d}z'_n$; we have for the i-th canonical inclusion $\jmath_i \colon A \to A^{\times n}$ that $\mathrm{d}\jmath_i = \mathrm{pr}_i$ is the i-th projection map. Since $A \otimes \Gamma_n$ is a fiber of the summation map, we have $\Omega^1_{A\otimes\Gamma_n} \simeq \Omega^1_{A^{\times n}} / \langle \mathrm{im}(\mathrm{d}\Sigma) \rangle$. Using that each $\jmath_i$ splits the summation map $\Sigma \colon A^{\times n} \to A$, we see that $\mathrm{d}\Sigma(\mathrm{d}z) = \mathrm{d}z_1 + \cdots + \mathrm{d}z_n$ and $\mathrm{d}\Sigma(\mathrm{d}z') = \mathrm{d}z'_1 + \cdots + \mathrm{d}z'_n$, and hence

$$\Omega^1_{A\otimes\Gamma_n} \simeq (\mathcal{O}_{A\otimes\Gamma_n} \otimes \Gamma_n^\vee)^{\oplus 2} \simeq \Omega^1_A \otimes \Gamma_n^\vee,$$

where the *dual standard representation* $\Gamma_n^\vee$ of S_n is defined as the cokernel of the diagonal map $\mathbb{Z} \to \mathbb{Z}^n$, cf. Definition 4.1.3. Taking determinants yields $\omega_{A\otimes\Gamma_n} \simeq \mathcal{O}_{A\otimes\Gamma_n} \otimes \det(\Gamma_n^\vee)^2$. Finally $\det(\Gamma_n^\vee)$ is the sign representation of S_n, so its square is trivial, which means that we are able to find a nowhere vanishing global n-form ω which is S_n-invariant. Concretely, we could take

$$\omega = \mathrm{d}z_1 \wedge \mathrm{d}z'_1 \wedge \cdots \wedge \mathrm{d}z_{n-1} \wedge \mathrm{d}z'_{n-1} \equiv \mathrm{d}z_2 \wedge \mathrm{d}z'_2 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}z'_n, \tag{1.1.2}$$

using the relations $\mathrm{d}\Sigma = 0$. We conclude as in the proof of Proposition 1.1.8. □

1.1.16. Remark. — The information gained in the proof of Proposition 1.1.15 tells us the following about the singularities of $X = (A \otimes \Gamma_n)/\mathrm{S}_n$. The singularities of X are (i) normal, since X is a categorical quotient of a normal variety, (ii) Cohen–Macaulay, since X is the quotient of a regular variety by a linearly reductive group, cf. [HR74, Main Thm.], (iii) Gorenstein, since ω_X is invertible, cf. [IshIS, Def. 6.2.1], (iv) canonical, since $\mathrm{HC}^* \omega_X \simeq \omega_{\mathrm{Kum}^{n-1}(A)}$, cf. [IshIS, Def. 6.2.4], (v) rational, since they are canonical, cf. [IshIS, Thm. 6.2.12].

1.1.17. Remark. — The generalized Kummer varieties of Example 1.1.10 are indeed a generalization of the Kummer surfaces of Example 1.1.6. For $n = 2$ we have an isomorphism

$$A \xrightarrow{\sim} A \otimes \Gamma_2, \quad a \mapsto (a, -a)$$

under which the permutation S_2-action on the right hand side corresponds to the action of $\mathbb{Z}/2\mathbb{Z} \simeq \mathrm{S}_2$ via the negation morphism $[-1] \colon A \to A$ on the left hand side. So we have an isomorphism

$$A/\langle[-1]\rangle \simeq (A \otimes \Gamma_2)/\mathrm{S}_2$$

of singular surfaces, and since the generalized Kummer variety $\mathrm{Kum}^1(A)$ described in ¶1.1.13 is a crepant resolution of the right hand side, it must be isomorphic to the Kummer surface described in Example 1.1.6 by ¶1.1.7.

(4) Not to be confused with the differential of some non-existent global 0-form.

1.1.18. — We list a couple of auxiliary facts about hyperkähler varieties and generalized Kummer varieties in particular.

(i) Besides the Hilbert schemes of points on a K3 surface and the generalized Kummer varieties, there are currently only two other deformation classes of hyperkähler varieties known, namely O'Grady's exceptional examples of dimension 6 and of dimension 10, cf. [OGr99; OGr03].

(ii) Strengthening the description of cohomology mentioned in the proof of Proposition 1.1.11, when $n \geq 3$, there is an isometry of integral Hodge structures

$$\mathrm{H}^2(\mathrm{Kum}^{n-1}(A), \mathbb{Z}) \simeq \mathrm{H}^2(A, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

where the left hand side is endowed with the Beauville–Bogomolov–Fujiki form (cf. [Bea83, Thm. 5], [Fuj87, Thm. 4.7]), and on the right hand side we have

$$\delta^2 = -2n \quad \text{and} \quad \delta \in \mathrm{H}^{1,1}(\mathrm{Kum}^{n-1}(A)),$$

cf. [Yos01, Lem. 4.10, Prop. 4.11]. In fact $2\delta = [E]$, where E is the exceptional divisor of the Hilbert–Chow morphism restricted to $\mathrm{Kum}^{n-1}(A)$. In particular the second Betti number is $\mathrm{b}_2((\mathrm{Kum}^{n-1}(A)) = \mathrm{b}_2(A) + 1 = 7$, and the Néron–Severi group is $\mathrm{NS}(\mathrm{Kum}^{n-1}(A)) \simeq \mathrm{NS}(A) \oplus \mathbb{Z}\delta$.

(iii) On the other hand we have $\mathrm{b}_2(\mathrm{Hilb}^n(S)) = \mathrm{b}_2(S) + 1 = 23$, when $n \geq 2$. So higher dimensional generalized Kummer varieties and Hilbert schemes of points on K3 surfaces are never birationally equivalent, since Betti numbers (even Hodge numbers) are birational invariants for smooth projective varieties with trivial canonical bundle, cf. [Bat99].

(iv) The Hodge numbers of generalized Kummer varieties are known in the form of a generating function for the Hodge polynomial, cf. [GS93, Cor. 1]. We provide the Hodge diamonds for a generalized Kummer fourfold as an example in Fig. 1.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 5 & & 1 & & \\
 & 0 & & 4 & & 4 & & 0 & \\
1 & & 5 & & 96 & & 5 & & 1 \\
 & 0 & & 4 & & 4 & & 0 & \\
 & & 1 & & 5 & & 1 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

FIGURE 1. Hodge diamond of a generalized Kummer fourfold

1.2. Abelian varieties and their polarizations

In this section we review the facts about abelian varieties which we will need in subsequent sections. In particular, we discuss the groups of homomorphisms between an abelian variety A and its dual $A^\vee$ when $\mathrm{End}(A) = \mathbb{Z}$, and we discuss the notion of dual polarizations. As general references we recommend and use [MumAV] and [EGM]. Only Proposition 1.2.27 is somewhat original.

1.2.1. Situation. — Let A and B be abelian varieties over a field $\Bbbk$. Let $g = \dim(A)$ denote the dimension of A; we are ultimately mainly interested in the case $g = 2$ of abelian surfaces.

***1.2.2. Definition* (Abelian varieties).** — A variety A over a field $\Bbbk$ together with the structure of an algebraic group on A over $\Bbbk$ is called an *abelian variety* if A is connected, smooth, and proper.

A *homomorphism* $f\colon A \to B$ between two abelian varieties A and B is a morphism $A \to B$ of varieties which is compatible with the respective group structures.[5]

1.2.3. Notation. — Denote by $\mathrm{Hom}(A, B) \subset \mathrm{Mor}_{\Bbbk}(A, B)$ the set of homomorphisms between A and B. When we want to stress that we use homomorphisms or automorphisms of abelian varieties, we write $\mathrm{Hom}_{\mathrm{AV}}(A, B)$ and $\mathrm{Aut}_{\mathrm{AV}}(A)$, respectively.

1.2.4. Proposition. — *Let A be an abelian variety over a field $\Bbbk$, then*

(i) *A is projective, and*

(ii) *the group law of A is commutative.*

Proof. — See [MumAV, §6, Application 1] and [MumAV, §4, Cor. 1], respectively. □

1.2.5. Remark. — Over the complex numbers $\Bbbk = \mathbb{C}$, the analytification A^{an} of an abelian variety A is a compact complex analytic group, so it is a complex torus, cf. [MumAV, §1, (2)].

So we have a homeomorphism $A^{\mathrm{an}} \simeq \mathbb{R}^{2g}/\mathbb{Z}^{2g} \simeq (\mathbb{S}^1)^{\times 2g}$, where $\mathbb{S}^1$ denotes the topological 1-sphere. One knows that $\mathrm{H}^\bullet(\mathbb{S}^1, \mathbb{Z}) \simeq \mathbb{Z}[x]/(x^2)$, so by the Künneth theorem we can write

$$\mathrm{H}^\bullet((\mathbb{S}^1)^{\times 2g}, \mathbb{Z}) \simeq \bigotimes_{i=1,\dots,2g} \mathrm{H}^\bullet(\mathbb{S}^1, \mathbb{Z})$$

as a tensor product of *graded commutative* rings. In particular, we see that

$$\mathrm{H}^k(A^{\mathrm{an}}, \mathbb{Z}) \simeq \textstyle\bigwedge^k \mathrm{H}^1(A^{\mathrm{an}}, \mathbb{Z}) \simeq \bigwedge^k(\mathrm{H}^1(\mathbb{S}^1, \mathbb{Z})^{\oplus 2g}).$$

1.2.6. Example. — An abelian variety of dimension $g = 1$ is just an elliptic curve. Let A and B be abelian varieties, then their product $A \times B$ is again an abelian variety. If $G \subset A$ is a closed subgroup scheme, then the (fppf) quotient A/G exists and is again an abelian variety, cf. [EGM, Thm. 4.39, Ex. 4.40].

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[5] In the definition of abelian varieties, smoothness is automatic in characteristic 0, cf. [SP, Tag 047N]. For a morphism of abelian varieties $f\colon A \to B$ to be a homomorphism, it suffices to require the compatibility $f(0_A) = 0_B$, where $0_A \in A(\Bbbk)$ and $0_B \in B(\Bbbk)$ denote the identity elements of the group structures on A and B, respectively, cf. ¶1.2.7.

Let C be a smooth, proper curve of genus g, then its Jacobian $\mathrm{Jac}(C)$ is the identity component $\mathrm{Pic}^0_{C/\Bbbk}$ of the Picard scheme $\mathrm{Pic}_{C/\Bbbk}$, and as such the "moduli space of degree 0 line bundles on C". The Jacobian $\mathrm{Jac}(C)$ is an abelian variety of dimension g, cf. [EGM, Ch. 14]. Coming back to elliptic curves, an elliptic curve E is canonically isomorphic to its own Jacobian $\mathrm{Jac}(E)$.

***1.2.7.* Morphisms of abelian varieties.** — For each $a \in A$ there is a *translation* morphism $\mathrm{t}_a \colon A \to A$ mapping $b \mapsto a + b$. If $f \colon A \to B$ is any morphism between two abelian varieties, then there exists $b \in B$ such that $\mathrm{t}_b \circ f$ becomes a homomorphism of abelian varieties, in fact $b = -f(0)$, cf. [MumAV, §4, Cor. 1].

A homomorphism $f \colon A \times B \to A' \times B'$ between products of abelian varieties can be written in matrix form as

$$f = \begin{pmatrix} f_1 \colon A \to A' & f_2 \colon B \to A' \\ f_3 \colon A \to B' & f_4 \colon B \to B' \end{pmatrix},$$

meaning $f(a, b) = (f_1(a) + f_2(b), f_3(a) + f_4(b))$ for $(a, b) \in A \times B$.

1.2.8. — The group $\mathrm{Pic}^0(A)$ of line bundles algebraically equivalent to $\mathcal{O}_A$ consists of line bundles $\mathcal{L} \in \mathrm{Pic}(A)$ such that $\mathrm{t}_a^*\mathcal{L} \simeq \mathcal{L}$ for every $a \in A$. The *dual abelian variety* $A^\vee$ is the fine moduli space parametrizing (rigidified) line bundles algebraically equivalent to $\mathcal{O}_A$. It is an abelian variety of dimension $\dim(A^\vee) = \dim(A)$, cf. [EGM, Thm. 6.18, Cor. 7.22] or [MumAV, §13]. In particular

$$A^\vee(\Bbbk) = \mathrm{Pic}^0(A),$$

and there exists the *Poincaré line bundle* $\mathcal{P} \in \mathrm{Pic}(A \times A^\vee)$ satisfying

$$\mathcal{P}_\alpha := \mathcal{P}|_{A \times \{\alpha\}} \simeq \mathcal{L}$$

exactly when $\alpha \in A^\vee(\Bbbk)$ corresponds to the line bundle $\mathcal{L}$, and the rigidification condition $\mathcal{P}|_{\{0\} \times A^\vee} \simeq \mathcal{O}_{A^\vee}$. The construction of the dual abelian variety is contravariantly functorial: Given a homomorphism $f \colon A \to B$, the *dual morphism* $f^\vee \colon B^\vee \to A^\vee$ is induced from the map pulling back line bundles on B to A. The dual morphism $f^\vee$ is again a homomorphism since $f^*\mathcal{O}_B \simeq \mathcal{O}_A$. Furthermore, if $g \colon A \to B$ is another homomorphism, we have the identity $(f + g)^\vee = f^\vee + g^\vee$, cf. [EGM, Cor. 7.17].

The variety $A^\vee$ deserves the name "dual abelian variety" because of the following: Using the description as a fine moduli space, we can specify a morphism in $\mathrm{Mor}(A, (A^\vee)^\vee)$ using a family of algebraically trivial line bundles on $A^\vee$ parametrized by A. Thus the Poincaré bundle $\mathcal{P}$ corresponds to the "evaluation" morphism

$$\mathrm{ev} \colon A \xrightarrow{\sim} A^{\vee\vee}, \tag{1.2.1}$$

which is an isomorphism, cf. [EGM, Thm. 7.9]. More precisely, to get a morphism, $\mathcal{P}$ should satisfy $\mathcal{P}|_{A \times \{0\}} \simeq \mathcal{O}_A$ and $\mathcal{P}|_{\{a\} \times A^\vee} \in \mathrm{Pic}^0(A^\vee)$ for every $a \in A$. The latter condition can be checked just at $a = 0$, cf. [EGM, Lem. 7.12], but $\mathcal{P}|_{\{0\} \times A^\vee} \simeq \mathcal{O}_{A^\vee}$ by 'rigidity', cf. [EGM, ¶6.2].

1.2.9. — Let A_1 and A_2 be abelian varieties, which are embedded into the product $A_1 \times A_2$ via the i-th coordinate embeddings $\iota_i \colon A_i \hookrightarrow A_1 \times A_2$. Then the homomorphism

$$(\iota_1^\vee, \iota_2^\vee) \colon (A_1 \times A_2)^\vee \to A_1^\vee \times A_2^\vee$$

is an isomorphism of abelian varieties, cf. [EGM, Exer. 6.2].

1.2.10. — Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian varieties. Dualizing it, we obtain again a short exact sequence of abelian varieties

$$0 \leftarrow A^\vee \leftarrow B^\vee \leftarrow C^\vee \leftarrow 0.$$

This can conceptually be explained by viewing (the rational points of) the dual abelian variety $A^\vee$ as the extension group $\mathrm{Ext}^1_{\mathbf{CAG}/\Bbbk}(A, \mathbb{G}_\mathrm{m})$ in the abelian category of commutative algebraic groups over $\Bbbk$, cf. [SerAGC, VII.3.16, Thm. 6] or [EGM, Thm. 8.9], and using the facts that $\mathrm{Ext}^2_{\mathbf{CAG}/\Bbbk}(A, \mathbb{G}_\mathrm{m}) = 0$ by [OorCGS, Prop. 12.3], and $\mathrm{Hom}_{\mathbf{CAG}/\Bbbk}(A, \mathbb{G}_\mathrm{m}) = 0$ since $\mathbb{G}_\mathrm{m}$ is affine and A is proper and connected.

In contrast to this, if $0 \to K \to A \to B \to 0$ is a short exact sequence where K is finite, then we obtain a short exact sequence

$$0 \to K^\mathrm{D} \to B^\vee \to A^\vee \to 0,$$

where K^D "=" $\mathrm{Hom}(K, \mathbb{G}_\mathrm{m})$ is the Cartier dual of the finite commutative algebraic group K, cf. [EGM, Thm. 3.22, Thm. 7.5].

1.2.11. — Given a line bundle $\mathcal{L} \in \mathrm{Pic}(A)$, there is a homomorphism $\varphi_\mathcal{L} \colon A \to A^\vee$ given on points by

$$\varphi_\mathcal{L}(a) = \mathrm{t}_a^* \mathcal{L} \otimes \mathcal{L}^\vee,$$

cf. [EGM, Cor. 2.10, §6.3]. So we have $\varphi_\mathcal{L} = 0$ exactly when $\mathcal{L} \in \mathrm{Pic}^0(A)$. In contrast, if $\mathcal{L}$ is ample, then $\varphi_\mathcal{L}$ has finite kernel, cf. [EGM, Lem. 2.19]. The homomorphism $\varphi_\mathcal{L} \colon A \to A^\vee$ is *symmetric*, which means that

$$\varphi_\mathcal{L} = \varphi_\mathcal{L}^\vee \circ \mathrm{ev}\,.$$

In fact, over an algebraically closed field $\overline{\Bbbk}$, this leads to an isomorphism

$$\mathrm{NS}(A) \coloneqq \mathrm{Pic}(A)/\,\mathrm{Pic}^0(A) \xrightarrow{\sim} \mathrm{Hom}_\mathrm{sym}(A, A^\vee)$$

of the Néron–Severi group of A with the group $\mathrm{Hom}_\mathrm{sym}(A, A^\vee)$ of symmetric homomorphisms, cf. [EGM, 7.26, Cor. 11.3]. Over an arbitrary field $\Bbbk$, this holds true when defining $\mathrm{NS}(A)$ as the $\Bbbk$-rational points of the fppf-quotient $\mathrm{Pic}_{A/\Bbbk} / \mathrm{Pic}^0_{A/\Bbbk}$ of the Picard scheme $\mathrm{Pic}_{A/\Bbbk}$ modulo its connected component $\mathrm{Pic}^0_{A/\Bbbk}$.

Regarding the interplay of the morphism $\varphi_\mathcal{L}$ with dual morphisms, we have for a homomorphism $f \colon A \to B$ and a line bundle $\mathcal{L} \in \mathrm{Pic}(B)$ a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi_{f^*\mathcal{L}}"'] & B \arrow[d, "\varphi_\mathcal{L}"] \\ A^\vee & B^\vee \arrow[l, "f^\vee"] \end{tikzcd} \tag{1.2.2}$$

which follows from the formula $f \circ \mathrm{t}_a = \mathrm{t}_{f(a)} \circ f$, cf. [EGM, Prop. 7.6].

***1.2.12.* Isogenies.** — A homomorphism $f\colon A \to B$ is called an *isogeny* if it is finite, flat[(6)], and surjective. When $\dim(A) = \dim(B)$, this is equivalent to asking that the kernel $\ker(f) \subset A$ is a finite algebraic group, or that f is surjective, cf. [EGM, Prop. 5.2]. Let f be an isogeny, then the *degree* $\deg(f)$ of f is defined as the rank of the kernel $\ker(f)$.[(7)] By ¶1.2.10, the dual morphism $f^\vee\colon B^\vee \to A^\vee$ is again an isogeny of degree $\deg(f^\vee) = \deg(f)$.

1.2.13. Example. — Assume $A \neq 0$. For each $n \in \mathbb{Z}$ we have an endomorphism $[n]\colon A \to A$ mapping $a \mapsto n \cdot a$. This provides a canonical inclusion $\mathbb{Z} \subset \operatorname{End}(A)$. Denote by

$$A[n] \coloneqq \ker([n])$$

the subgroup scheme of n-torsion points of A. Assume that $\operatorname{char}(\Bbbk) \nmid n$ for (1.2.3) and (1.2.4). For $n \neq 0$ we have that $[n]$ is an isogeny of degree

$$\#A[n] = \deg([n]) = n^{2g}, \tag{1.2.3}$$

cf. [MumAV, §6, App. 2–3]; the equality on the right hand side holds without any assumption on the characteristic of $\Bbbk$. In fact, the structure theorem of finite abelian groups leads to an isomorphism

$$A[n](\overline{\Bbbk}) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}, \tag{1.2.4}$$

where the left hand side is the group of geometric n-torsion points, cf. [EGM, Cor. 5.11]. By the surjectivity of $[n]$, we see that the group $A(\overline{\Bbbk})$ of geometric points of A is n-divisible for each $0 \neq n \in \mathbb{Z}$, i.e. for each $a \in A(\overline{\Bbbk})$ there exists some $a' \in A(\overline{\Bbbk})$ such that $na' = a$. Lastly, we have the identity $[n]^\vee = [n]$ since taking duals commutes with taking sums of homomorphisms.

1.2.14. — The map $[n]\colon A \to A$ is in fact a *separable isogeny*, i.e. étale, if $\operatorname{char}(\Bbbk) \nmid n$, cf. [EGM, Prop. 5.6]. If furthermore $\Bbbk$ is algebraically closed, we deduce from Example 1.2.13 that it is a Galois cover with respect to the group $A[n](\Bbbk)$, i.e. we have a fibre product square

$$\begin{tikzcd} A[n](\Bbbk) \times A \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{act}"'] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d, "{[n]}"] \\ A \arrow[r, "{[n]}"] & A, \end{tikzcd} \tag{1.2.5}$$

where $A[n](\Bbbk)$ acts on A by translations, cf. [SzaGCF, Prop. 5.3.16]. Let $\mathcal{L} \in \operatorname{Pic}(A)$ be a line bundle, then we have

$$[n]^*[n]_*\mathcal{L} \simeq \mathrm{act}_* \mathrm{pr}_2^*\mathcal{L} \simeq \bigoplus_{a \in A[n](\Bbbk)} \mathrm{t}_{a,*}\mathcal{L} \simeq \bigoplus_{a' \in A[n](\Bbbk)} \mathrm{t}_{a'}^*\mathcal{L}$$

by affine base change [SP, Tag 02KG] along the square (1.2.5).

---

(6) In fact, given the other conditions, flatness is automatic.

(7) The degree $\deg(f)$ is equal to the degree of the field extension $f^*\colon \mathrm{K}(B) \hookrightarrow \mathrm{K}(A)$, while the separable degree of this extension is equal to the order of $\ker(f)$, cf. [MumAV, §6, App. 3].

1.2.15. — Let $\mathcal{L} \in \mathrm{Pic}(A)$ be a line bundle. Then one obtains as a corollary of the Theorem of the Cube, that

$$[n]^*\mathcal{L} \simeq \mathcal{L}^{\otimes n(n+1)/2} \otimes [-1]^*\mathcal{L}^{\otimes n(n-1)/2}, \tag{1.2.6}$$

cf. [EGM, Cor. 2.12]. If $\mathcal{L}$ is *symmetric*, i.e. $[-1]^*\mathcal{L} \simeq \mathcal{L}$, then (1.2.6) becomes $[n]^*\mathcal{L} \simeq \mathcal{L}^{\otimes n^2}$, and if $\mathcal{L}$ is *anti-symmetric*, i.e. $[-1]^*\mathcal{L} \simeq \mathcal{L}^\vee$, then (1.2.6) becomes $[n]^*\mathcal{L} \simeq \mathcal{L}^{\otimes n}$. Considering the tensor decomposition of $\mathcal{L}^{\otimes 2}$ into the symmetric line bundle $\mathcal{L} \otimes [-1]^*\mathcal{L}$ and the anti-symmetric line bundle $\mathcal{L} \otimes [-1]^*\mathcal{L}^\vee$, the latter being algebraically equivalent to $\mathcal{O}_A$ by [EGM, Cor. 7.23], we calculate in the Néron–Severi group $\mathrm{NS}(A)$ that

$$2[n]^*[\mathcal{L}] = [n]^*[\mathcal{L}^{\otimes 2}] = [n]^*[\mathcal{L} \otimes [-1]^*\mathcal{L}] = n^2[\mathcal{L} \otimes [-1]^*\mathcal{L}] = 2n^2[\mathcal{L}] \in \mathrm{NS}(A).$$

So, since $\mathrm{NS}(A)$ is torsion free (cf. [EGM, Cor. 7.25]), and after base change to $\overline{\Bbbk}$ every class in $\mathrm{NS}(A_{\overline{\Bbbk}})$ is represented by a line bundle $\mathcal{L} \in \mathrm{Pic}(A_{\overline{\Bbbk}})$, we see that pullback along $[n]$ acts on Néron–Severi groups as in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}^0(A) \arrow[r] \arrow[d, "\cdot n"] & \mathrm{Pic}(A) \arrow[r] \arrow[d, "{[n]^*}"] & \mathrm{NS}(A) \arrow[r, dashed] \arrow[d, "\cdot n^2"] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(A) \arrow[r] & \mathrm{Pic}(A) \arrow[r] & \mathrm{NS}(A) \arrow[r, dashed] & 0.
\end{tikzcd}$$

***1.2.16. Definition* (Exponents of isogenies).** — The *exponent* $\mathrm{e}(f)$ of an isogeny $f\colon A \to B$ is defined as the smallest positive natural number e such that $\ker(f) \subset A[e]$. It is clear that the exponent divides the degree, $\mathrm{e}(f) \mid \deg(f)$, since $\ker(f) \subset A[\deg(f)]$, cf. [EGM, Exer. 4.4].

***1.2.17. Definition* (Polarizations).** — A homomorphism $\lambda\colon A \to A^\vee$ is called a *polarization* if λ is

(i) an isogeny, i.e. surjective with finite kernel,
(ii) symmetric, i.e. $\lambda = \lambda^\vee \circ \mathrm{ev}$, and
(iii) the pullback $(\mathrm{id}, \lambda)^*\mathcal{P}$ of the Poincaré bundle along the morphism $(\mathrm{id}, \lambda)\colon A \to A \times A^\vee$ is ample.

We call A *principally polarizable* if it admits a polarization λ of degree $\deg(\lambda) = 1$, i.e. if λ is an isomorphism $A \xrightarrow{\sim} A^\vee$; in this case λ is called a principal polarization.

1.2.18. Remark. — A polarization $\lambda\colon A \to A^\vee$ is of the form $\lambda = \varphi_{\mathcal{L}}$ for some ample line bundle $\mathcal{L} \in \mathrm{Pic}(A)$ after base-change to some finite field extension of $\Bbbk$, cf. [EGM, Cor. 11.5]. By Riemann–Roch, cf. [EGM, Thm. 9.11], we have

$$\deg(\varphi_{\mathcal{L}}) = \chi(\mathcal{L})^2 \quad \text{and} \quad \chi(\mathcal{L}) = \mathrm{c}_1(\mathcal{L})^g/g!,$$

so the degree of the polarized variety $(A, \mathcal{L})$ equals $g!\sqrt{\deg(\lambda)}$. For the case $g = 2$, which will interest us the most, we thus have for some $d \in \mathbb{N}$ that

$$\deg(\lambda) = d^2 \quad \text{and} \quad \deg(A, \mathcal{L}) = 2d.$$

1.2.19. Example. — Let C be a smooth, proper curve, then its Jacobian $\mathrm{Jac}(C)$ is principally polarized, where the isomorphism $\lambda\colon \mathrm{Jac}(C) \xrightarrow{\sim} \mathrm{Jac}(C)^\vee$ is induced by the line bundle $\mathcal{O}_C(\Theta)$, where Θ is a so called "theta divisor" on $\mathrm{Jac}(C)$, cf. [EGM, Thm. 14.23].

Following [EGM, 11.24], to see that there exist non-principally polarizable abelian varieties, one starts with a principally polarized abelian variety A over $\overline{\Bbbk}$ of dimension $g \geq 2$ with $\mathrm{End}(A) = \mathbb{Z}$. The Jacobians $\mathrm{Jac}(C)$ associated to certain hyperelliptic curves C provide such abelian varieties in characteristic zero, cf. [Mor77], and in positive characteristic under further restrictions, in particular $\overline{\Bbbk} \neq \overline{\mathbb{F}}_p$, cf. [Zar18, Thm. 1.1]. Suppose $\lambda\colon A \to A^\vee$ is a principal polarization. Any other morphism $A \to A^\vee$ is of the form $n\lambda$, for some $n \in \mathbb{Z}$, and has degree n^{2g}, cf. Proposition 1.2.25 and Example 1.2.13. Let $\ell \neq \mathrm{char}(\overline{\Bbbk})$ be a prime number and let $G \subset A$ be a subgroup scheme of order ℓ, as may be found inside $A[\ell] \simeq (\mathbb{Z}/\ell\mathbb{Z})^{2g}$. Now consider the quotient

$$q\colon A \to A/G,$$

and let $\mu\colon A/G \to (A/G)^\vee$ be any polarization. Then the homomorphism

$$q^\vee \circ \mu \circ q\colon A \to A/G \to (A/G)^\vee \to A^\vee$$

has degree $\ell^2 \cdot \deg(\mu)$, which is not of the required form n^{2g} if $\deg(\mu) = 1$.

1.2.20. — Let $\lambda\colon A \to A^\vee$ be a polarization of an abelian variety over an algebraically closed field $\Bbbk = \overline{\Bbbk}$, and assume that λ is a *separable polarization*, i.e. $\mathrm{char}(\Bbbk) \nmid \deg(\lambda)$. Following [Mum66, §1], one can endow the kernel $\ker(\lambda)$ with a (multiplicatively written) non-degenerate alternating form, which implies that the elementary divisors of the finite group $\ker(\lambda)$ appear in pairs. So we can write

$$\ker(\lambda) \simeq (\mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_g\mathbb{Z})^2,$$

where $d_i \geq 1$ such that $d_i \mid d_{i+1}$. Then one calls $(d_1, \ldots, d_g)$ the *type of the polarization* λ. Note that we have indeed no more than $2g$ elementary divisors, since $\ker(\lambda) \subset A[\deg(\lambda)] \simeq (\mathbb{Z}/\deg(\lambda)\mathbb{Z})^{2g}$. Thus a polarization λ of type $(d_1, \ldots, d_g)$ has degree and exponent

$$\deg(\lambda) = (\prod_{i=1}^{g} d_i)^2 \quad \text{and} \quad \mathrm{e}(\lambda) = d_g.$$

The type of a polarization $\lambda\colon A \to A^\vee$ over $\Bbbk$ is defined as the type of the polarization $\lambda_{\overline{\Bbbk}}$ arising by base change to $\overline{\Bbbk}$.

***1.2.21.* Dual isogenies and dual polarizations.** — Let $f\colon A \to B$ be an isogeny of degree $d = \deg(f)$. Then there exists a unique isogeny $\widehat{f}\colon B \to A$, called the *dual isogeny*, satisfying $f \circ \widehat{f} = [d]$ and $\widehat{f} \circ f = [d]$, which is constructed as follows. Since $\ker(f) \subset A[d]$, cf. [EGM, Exer. 4.4], we can factor the homomorphism $[d]\colon A \to A$ over the homomorphism f, diagramatically

$$\begin{tikzcd}
\ker(f) \arrow[r, hook] \arrow[dr, "0"'] & A \arrow[r, "{[d]}"] \arrow[d, "f"] & A \\
& B. \arrow[ur, dashed, "\widehat{f}"'] &
\end{tikzcd}$$

Now let $\lambda\colon A \to A^\vee$ be an isogeny of exponent $\mathrm{e}(\lambda)$. By the same argument as above, there exists and isogeny $\lambda^{\mathrm{D}}\colon A^\vee \to A$ satisfying

$$\lambda \circ \lambda^{\mathrm{D}} = [\mathrm{e}(\lambda)] \quad \text{and} \quad \lambda^{\mathrm{D}} \circ \lambda = [\mathrm{e}(\lambda)].$$

Following [BL03, §2], if $\lambda\colon A \to A^\vee$ is a separable polarization of type $(d_1, \ldots, d_g)$, then the *dual polarization* is defined as

$$\lambda^\delta := d_1 \lambda^{\mathrm{D}},$$

and satisfies $(\lambda^\delta)^\delta = \lambda$, cf. [BL03, Prop. 2.3].

1.2.22. Remark. — As a warning, the *dual polarization* $\lambda^\delta\colon A^\vee \to A$ (or λ^{D}) should neither be confused with the *dual isogeny* $\widehat{\lambda}\colon A^\vee \to A$, nor with the dual homomorphism $\lambda^\vee\colon A^{\vee\vee} \to A^\vee$.

1.2.23. Theorem. — *Let $\lambda\colon A \to A^\vee$ be a separable polarization of type $(d_1, \ldots, d_g)$. Then $\lambda^{\mathrm{D}}\colon A^\vee \to A \simeq A^{\vee\vee}$ is a separable polarization of type*

$$\mathrm{type}(\lambda^{\mathrm{D}}) = \left(1, \frac{d_g}{d_{g-1}}, \ldots, \frac{d_g}{d_1}\right).$$

In particular $\lambda^\delta\colon A^\vee \to A \simeq A^{\vee\vee}$ is a separable polarization as well, and its type is

$$\mathrm{type}(\lambda^\delta) = \left(d_1, \frac{d_1 d_g}{d_{g-1}}, \ldots, \frac{d_1 d_g}{d_2}, d_g\right).$$

Proof. — This is exactly [BL03, Thm. 2.1] and [BL03, Prop. 2.2]. We reproduce the details here. Without loss of generality, we base change to an algebraic closure $\overline{\Bbbk}$. We already know that λ^{D} is an isogeny. We show that λ^{D} is symmetric, i.e. $(\lambda^{\mathrm{D}})^\vee = \mathrm{ev} \circ \lambda^{\mathrm{D}}$. Dualizing the equation $\lambda \circ \lambda^{\mathrm{D}} = [e]$ yields $(\lambda^{\mathrm{D}})^\vee \circ \lambda^\vee = [e]$. We know λ is symmetric, i.e. $\lambda = \lambda^\vee \circ \mathrm{ev}$, so by substituting we obtain $(\lambda^{\mathrm{D}})^\vee \circ \lambda = [e] \circ \mathrm{ev}$. On the other hand, from $\lambda^{\mathrm{D}} \circ \lambda = [e]$ we get

$$\mathrm{ev} \circ \lambda^{\mathrm{D}} \circ \lambda = \mathrm{ev} \circ [e] = [e] \circ \mathrm{ev}\,.$$

Comparing these equations and cancelling λ, since it is surjective, yields $(\lambda^{\mathrm{D}})^\vee = \mathrm{ev} \circ \lambda^{\mathrm{D}}$ as desired.

By symmetry, we know that we can write $\lambda^{\mathrm{D}} = \varphi_{\mathcal{M}}$ for some line bundle $\mathcal{M} \in \mathrm{Pic}(A^\vee)$, and we need to check that $\mathcal{M}$ is ample. We have the chain of equalities

$$\varphi_{\lambda^*\mathcal{M}} = \lambda^\vee \circ \varphi_{\mathcal{M}} \circ \lambda = \lambda^\vee \circ \lambda^{\mathrm{D}} \circ \lambda = \lambda^\vee \circ \mathrm{ev} \circ \lambda^{\mathrm{D}} \circ \lambda = \lambda \circ [e],$$

where we used (1.2.2), the implicit evaluation morphism in λ^{D}, the symmetry of λ, and the property $\lambda^{\mathrm{D}} \circ \lambda = [e]$. Since λ is a polarization, $\lambda \circ [e] = e\lambda$ is a polarization as well, which implies that $\lambda^*\mathcal{M}$ is ample. Since λ is a finite surjective morphism of proper schemes, this implies that $\mathcal{M}$ is already ample, cf. [HarAS, Prop. I.4.4].

Since λ is separable, we know that $\mathrm{char}(\Bbbk)$ is not a divisor of $\mathrm{e}(\lambda)$, so $\lambda \circ \lambda^{\mathrm{D}} = [\mathrm{e}(\lambda)]$ implies that λ^{D} is separable as well.

Regarding the type of λ^{D}, by the construction of λ^{D} in ¶1.2.21 and the definition of the type of a polarization, we have

$$\begin{aligned}\ker(\lambda^{\mathrm{D}}) &\simeq \ker([e])/\ker(\lambda)\\ &\simeq (\mathbb{Z}/d_g\mathbb{Z})^{2g}/(\mathbb{Z}/d_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/d_g\mathbb{Z})^2\\ &\simeq (\mathbb{Z}/\tfrac{d_g}{d_1}\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/\tfrac{d_g}{d_{g-1}}\mathbb{Z})^2.\end{aligned}$$

□

***1.2.24.* Simple abelian varieties.** — An abelian variety A is *simple* if it is non-zero and its only abelian subvarieties are 0 and A itself. An abelian variety A is simple if and only if $\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ is a division algebra, cf. [MumAV, §19, Cor. 2]; in particular, if $\mathrm{End}(A)=\mathbb{Z}$, then A is simple.

1.2.25. Proposition. —

(i) *There exist (non-canonical) injections of groups* $\mathrm{Hom}(A,A^\vee)\hookrightarrow\mathrm{End}(A)$ *and* $\mathrm{End}(A)\hookrightarrow\mathrm{Hom}(A,A^\vee)$.

(ii) *We have an isomorphism of algebras* $\mathrm{End}(A)\simeq\mathrm{End}(A^\vee)^{\mathrm{op}}$. *So if* $\mathrm{End}(A)=\mathbb{Z}$, *then* $\mathrm{End}(A^\vee)=\mathbb{Z}$, *and* $\mathrm{Hom}(A,A^\vee)\simeq\mathbb{Z}$, *and* $\mathrm{NS}(A)\simeq\mathbb{Z}$.

Proof. — (i) There exist isogenies $\lambda\colon A\to A^\vee$ and $\lambda'\colon A^\vee\to A$, e.g. a polarization and its dual isogeny. Consider the group homomorphism

$$\psi\colon\ \mathrm{Hom}(A,A^\vee)\to\mathrm{End}(A),\quad f\mapsto\lambda'\circ f.$$

If $\psi(f)=0$, then $\mathrm{im}(f)\subset\ker(\lambda')$. Since A is connected and $\ker(\lambda')$ is finite, we see that f is constant, with value 0 since $f(0)=0$. Similarly, we see that $\mathrm{End}(A)\to\mathrm{Hom}(A,A^\vee)$, $f\mapsto\lambda\circ f$ is injective.

(ii) Consider the group homomorphism

$$\delta_A\colon\ \mathrm{End}(A)\to\mathrm{End}(A^\vee),\quad f\mapsto f^\vee.$$

The composition $\delta_{A^\vee}\circ\delta_A$ is the identity when we identify A with $A^{\vee\vee}$ using the isomorphism $\mathrm{ev}\colon A\to A^{\vee\vee}$. Indeed, for every $f\in\mathrm{End}(A)$, we have $f=\mathrm{ev}^{-1}\circ f^{\vee\vee}\circ\mathrm{ev}$, as is readily checked using the functor of points of the dual abelian variety, cf. [EGM, §6.2, Def. 6.19]. So we conclude $\mathrm{End}(A)\simeq\mathrm{End}(A^\vee)$.

Finally, by (i), $\mathrm{Hom}(A,A^\vee)$ is a subgroup of $\mathrm{End}(A)=\mathbb{Z}$, so it is isomorphic to $\mathbb{Z}$ itself. Similarly, $\mathrm{NS}(A)$ identifies with the subgroup of symmetric homomorphisms in $\mathrm{Hom}(A,A^\vee)$, so also $\mathrm{NS}(A)\simeq\mathbb{Z}$. □

To exercise the definitions further we prove the following elementary statement.

1.2.26. Proposition. — *If* $\mathrm{Hom}(A,A^\vee)=\mathbb{Z}\cdot\lambda_0$, *then* λ_0 *or* $-\lambda_0$ *is a polarization.*

Proof. — We know that there exists some polarization $\lambda\colon A\to A^\vee$. By assumption we can write $\lambda=n\cdot\lambda_0$ for some $n\in\mathbb{Z}$. Let us assume that $n\geq 1$, otherwise replace λ_0 by $-\lambda_0$. First, we check that λ_0 is an isogeny. Since λ is surjective, we find that $\lambda_0\circ[n]=\lambda$ is surjective as well, which in turn shows that λ_0 is surjective.

Second, we check that λ_0 is symmetric. We know by assumption that $[n]\circ\lambda_0=\lambda$ is symmetric, i.e. $[n]\circ\lambda_0=([n]\circ\lambda_0)^\vee\circ\mathrm{ev}$. Substituting $([n]\circ\lambda_0)^\vee=\lambda_0^\vee\circ[n]^\vee=\lambda_0^\vee\circ[n]$, we see that $\lambda_0 \circ [n] = \lambda_0^\vee \circ \mathrm{ev} \circ [n]$. Since $[n]$ is surjective, this implies $\lambda_0 = \lambda_0^\vee \circ \mathrm{ev}$ as desired.

Third, we check that $(\mathrm{id}, \lambda_0)^*\mathcal{P}$ is an ample line bundle on A. We have

$$(\mathrm{id}, \lambda)^*\mathcal{P} = (\mathrm{id}, n\lambda_0)^*\mathcal{P} = (\mathrm{id}, \lambda_0)^*(\mathrm{id} \times [n])^*\mathcal{P} \simeq (\mathrm{id}, \lambda_0)^*(\mathcal{P}^{\otimes n}) \simeq ((\mathrm{id}, \lambda_0)^*\mathcal{P})^{\otimes n},$$

by [EGM, Lem. 7.16] using $\mathcal{P}|_{\{0\}\times A^\vee} \simeq \mathcal{O}_{A^\vee}$. Finally $(\mathrm{id}, \lambda_0)^*\mathcal{P}$ is ample since some positive power of it is ample by assumption. □

1.2.27. Proposition. — *Assume $g = 2$ and $\mathrm{Hom}(A, A^\vee) = \mathbb{Z} \cdot \lambda_0$ for some separable polarization λ_0, then*

(i) $\mathrm{Hom}(A^\vee, A) = \mathbb{Z} \cdot \lambda_0^\delta$, *where λ_0^δ denotes the dual polarization of λ_0, and*

(ii) $\lambda_0^\delta \circ \lambda_0 = [\mathrm{e}(\lambda_0)]$, *as well as $\lambda_0 \circ \lambda_0^\delta = [\mathrm{e}(\lambda_0)]$, with $\mathrm{e}(\lambda_0)^2 = \deg(\lambda_0)$.*

Proof. — (i) In order to keep the formulas a bit more transparent, we consider g to be arbitrary in the first part of this argument. Let $(d_1, \dots, d_g)$ be the type of the polarization λ_0, then by Theorem 1.2.23 the type of λ_0^{D} is

$$(1, d_g/d_{g-1}, \dots, d_g/d_1),$$

and the type of $(\lambda_0^{\mathrm{D}})^{\mathrm{D}}$ is

$$(1, d_2/d_1, \dots, d_g/d_1).$$

This allows us to compute the degrees

$$\begin{aligned}
\deg(\lambda_0) &= d_1^2 \dots d_g^2,\\
\deg(\lambda_0^{\mathrm{D}}) &= (d_g \dots d_g)^2/(d_g \dots d_1)^2 = d_g^{2g}/\deg(\lambda_0),\\
\deg((\lambda_0^{\mathrm{D}})^{\mathrm{D}}) &= (d_1 \dots d_g)^2/(d_1 \dots d_1)^2 = \deg(\lambda_0)/d_1^{2g}.
\end{aligned}$$

Now, $\mathrm{Hom}(A, A^\vee) = \mathbb{Z} \cdot \lambda_0$ implies that $\deg(\lambda_0)$ divides $\deg((\lambda_0^D)^D) = \deg(\lambda_0)/d_1^{2g}$, which shows that $d_1 = 1$. Using $g = 2$, we see that

$$\deg(\lambda_0^{\mathrm{D}}) = d_2^4/d_2^2 = \deg(\lambda_0).$$

Writing $\mathrm{Hom}(A^\vee, A) = \mathbb{Z} \cdot \widetilde{\lambda}$, and applying the above arguments to $\widetilde{\lambda}$, yields the equality $\deg(\widetilde{\lambda}^{\mathrm{D}}) = \deg(\widetilde{\lambda})$. So $\deg(\widetilde{\lambda}) \geq \deg(\lambda_0^{\mathrm{D}})$, otherwise $\deg(\widetilde{\lambda}) < \deg(\lambda_0^{\mathrm{D}})$ would imply $\deg(\widetilde{\lambda}^{\mathrm{D}}) < \deg(\lambda_0)$, contradicting the minimality of $\deg(\lambda_0)$. In conclusion, since λ_0^{D} is already a multiple of $\widetilde{\lambda}$, we get $\widetilde{\lambda} = \lambda_0^{\mathrm{D}}$.

(ii) Note that the arguments above show that $\lambda_0^\delta = \lambda_0^{\mathrm{D}}$ and $\mathrm{e}(\lambda_0)^2 = d_2^2 = \deg(\lambda_0)$. □

1.3. Semi-homogeneous and unipotent vector bundles

We briefly exposit Mukai's theory of semi-homogeneous vector bundles on abelian varieties and state a few facts concerning them that will be useful later on in §6.2. See [Muk78] for reference and details. Everything, except the splitting criterion in Proposition 1.3.12, can be found in the works of Mukai.

1.3.1. Situation. — Let A be an abelian variety of dimension $\dim(A) = g$ over an algebraically closed field $\Bbbk = \overline{\Bbbk}$.

***1.3.2. Definition* (Semi-homogeneous vector bundle).** — A vector bundle $\mathcal{E}$ on A is called *semi-homogeneous* if for every $a \in A$ there exists some line bundle $\mathcal{L}_a \in \mathrm{Pic}^0(A)$ such that

$$\mathrm{t}_a^*\mathcal{E} \simeq \mathcal{E} \otimes \mathcal{L}_a.$$

If each $\mathcal{L}_a$ is the trivial line bundle $\mathcal{O}_A$, then one says $\mathcal{E}$ is *homogeneous*. The vector bundle $\mathcal{E}$ is called *simple* if $\mathrm{End}(\mathcal{E}) = \Bbbk$.

1.3.3. Example. — Every line bundle $\mathcal{L} \in \mathrm{Pic}(A)$ is semi-homogeneous, indeed

$$\mathrm{t}_a^*\mathcal{L} \simeq \mathcal{L} \otimes \mathcal{P}_{\varphi_{\mathcal{L}}(a)},$$

and $\mathcal{L}$ is homogeneous if and only if $\mathcal{L} \in \mathrm{Pic}^0(A)$, i.e. $\mathcal{L}$ is algebraically trivial.

If $\mathcal{E}$ is semi-homogeneous, then the vector bundle $\mathcal{E}nd(\mathcal{E})$ is homogeneous.

Let $f\colon A' \to A$ be an isogeny and $\mathcal{L}' \in \mathrm{Pic}(A')$ a line bundle on A', then $f_*\mathcal{L}'$ is a semi-homogeneous vector bundle on A, cf. [Muk78, Prop. 5.4]. Conversely, every simple semi-homogeneous vector bundle $\mathcal{E}$ on A is of the above form $f_*\mathcal{L}'$, cf. [Muk78, Thm. 5.8]. But this construction does not alway lead to interesting bundles, for example if $\mathrm{char}(\Bbbk) \nmid n$, then

$$[n]_*\mathcal{O}_A \simeq \bigoplus_{\alpha \in A^\vee[n]} \mathcal{P}_\alpha,$$

and, by the projection formula, we have a similar decomposition for the push forward of a line bundle of the form $[n]^*\mathcal{L}$, cf. [Muk78, Cor. 4.22].

1.3.4. Definition. —

(i) Let $\mathcal{E} \neq 0$ be a vector bundle on A. Define the *slope* $\mu(\mathcal{E})$ of $\mathcal{E}$ as

$$\mu(\mathcal{E}) \coloneqq [\det(\mathcal{E})] \otimes \frac{1}{\mathrm{rk}(\mathcal{E})} \in \mathrm{NS}(A) \otimes \mathbb{Q}.$$

(ii) Given an element $\mu = [\mathcal{L}] \otimes \frac{1}{\ell} \in \mathrm{NS}(A) \otimes \mathbb{Q}$ (with $\ell \in \mathbb{Z}$), one defines

$$\Phi_\mu \coloneqq \mathrm{im}(([\ell], \varphi_{\mathcal{L}})\colon A \to A \times A^\vee).$$

Denote the projection onto the first factor by $\mathrm{pr}_1\colon \Phi_\mu \hookrightarrow A \times A^\vee \to A$, and denote its kernel by

$$\Sigma_\mu \coloneqq \ker(\mathrm{pr}_1\colon \Phi_\mu \to A).$$

1.3.5. Remark. — Note that the image in the definition of Φ_μ depends only on μ instead of $\mathcal{L}$ and ℓ, essentially due to the fact that the maps $[n]\colon A \to A$ are surjective.

1.3.6. — If $\mathcal{E}$ is a simple semi-homogeneous vector bundle of slope μ, then $(a, \alpha) \in \Phi_\mu$ if and only if $\mathrm{t}_a^*\mathcal{E} \simeq \mathcal{E} \otimes \mathcal{P}_\alpha$, where $\mathcal{P} \in \mathrm{Pic}(A \times A^\vee)$ is the Poincaré bundle, cf. [Muk78, Prop. 7.7], [Orl02, Lem. 4.9].

We can view $\Sigma_\mu \subset A \times A^\vee$ as a subgroup of the second factor $A^\vee$. Then the description of Φ_μ above means that

$$\Sigma_\mu = \{\alpha \in A^\vee \mid \mathcal{E} \simeq \mathcal{E} \otimes \mathcal{P}_\alpha\}.$$

On the other hand, writing $\mu = [\mathcal{L}] \otimes \frac{1}{\ell}$, the definition of Φ_μ tells us that we have an equality

$$\Sigma_\mu = \{\varphi_{\mathcal{L}}(a) \mid a \in A, \ell a = 0\}.$$

Thus we have a short exact sequence

$$0 \to \ker(\varphi_{\mathcal{L}})[\ell] \to A[\ell] \xrightarrow{\varphi_{\mathcal{L}}} \Sigma_\mu \to 0,$$

and we see that $\Sigma_\mu \subset A^\vee[\ell]$, cf. [Muk78, Cor. 7.8].

The next proposition is concerned with the construction of (simple) semi-homogeneous vector bundles of a given slope.

1.3.7. Proposition. — *Let $\mu = [\mathcal{L}] \otimes \frac{1}{\ell} \in \mathrm{NS}(A) \otimes \mathbb{Q}$ with $\ell > 0$.*

(i) *The sheaf $\mathcal{F} := [\ell]_*(\mathcal{L}^{\otimes \ell})$ is a semi-homogeneous vector bundle on A with slope $\mu(\mathcal{F}) = \mu$.*

(ii) *Let $\mathcal{F}$ be a semi-homogeneous vector bundle on A of slope $\mu(\mathcal{F}) = \mu$, then there exists a Jordan–Hölder filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}$$

such that each $\mathcal{E}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ is a simple semi-homogeneous vector bundle of slope $\mu(\mathcal{E}_i) = \mu$. Such a filtration is not necessarily unique but the associated multiset $\{\{\mathcal{E}_i\}\}$ of grades pieces is unique.

Proof. — See [Muk78, Prop. 6.22, Prop. 6.15]. We spell out the details for (i) in the case of characteristic $\mathrm{char}(\Bbbk) = 0$, following Mukai. First, in greater generality, let $f\colon A' \to A$ be an isogeny and let $\mathcal{E}$ be a semi-homogeneous vector bundle on A'. Since an isogeny is finite locally free(8), we see that $f_*\mathcal{E}$ is again a vector bundle, of rank $\mathrm{rk}(f_*\mathcal{E}) = \deg(f)\,\mathrm{rk}(\mathcal{E})$.

Second, let $a \in A$ and pick $a' \in A'$ such that $f(a') = a$. Then there exist some line bundle $\mathcal{L}'_{a'} \in \mathrm{Pic}^0(A')$ such that $\mathrm{t}_{a'}^*\mathcal{E} \simeq \mathcal{E} \otimes \mathcal{L}'_{a'}$. Since f is an isogeny, we know that $f^*\colon \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(A')$ is surjective, cf. ¶1.2.10, so we can pick a line bundle $\mathcal{L}_a \in \mathrm{Pic}^0(A)$ with $f^*\mathcal{L}_a \simeq \mathcal{L}'_{a'}$. By base change and the projection formula, we calculate

$$\mathrm{t}_a^* f_*\mathcal{E} \simeq f_*\mathrm{t}_{a'}^*\mathcal{E} \simeq f_*(\mathcal{E} \otimes \mathcal{L}'_{a'}) \simeq f_*(\mathcal{E} \otimes f^*\mathcal{L}_a) \simeq f_*\mathcal{E} \otimes \mathcal{L}_a.$$

(8) A morphism $f\colon X \to Y$ is *finite locally free* if it is finite, flat, and locally of finite presentation. This is equivalent to being affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module, cf. [SP, Tag 02K9].

Third, we calculate

$$f^*(\det(f_*\mathcal{E})) \simeq \det(f^* f_*\mathcal{E}) \simeq \det(\bigoplus_{a'\in\ker(f)} \mathrm{t}_{a'}^*\mathcal{E})$$
$$\simeq \bigotimes_{a'\in\ker(f)} \det(\mathcal{E}\otimes\mathcal{L}'_{a'}) \equiv \det(\mathcal{E})^{\otimes \deg(f)},$$

where the last congruence "$\equiv$" is modulo $\mathrm{Pic}^0(A')$, i.e. "$\equiv$" can be viewed as equality inside $\mathrm{NS}(A')$.

Now specializing to $f = [\ell]$, and $\mathcal{E} = \mathcal{L}^{\otimes\ell}$, and $\mathcal{F} = [\ell]_*\mathcal{L}^{\otimes\ell}$, we get the equalities

$$[\ell]^*[\det(\mathcal{F})] \equiv [(\mathcal{L}^{\otimes\ell})^{\otimes\ell^{2g}}] \equiv \ell^{2g+1}[\mathcal{L}] \in \mathrm{NS}(A).$$

We also have $[\ell]^*[\det(\mathcal{F})] \equiv \ell^2[\det(\mathcal{F})]$ by ¶1.2.15, so $[\det(\mathcal{F})] \equiv \ell^{2g-1}[\mathcal{L}]$. Finally,

$$\mu(\mathcal{F}) \overset{\text{def}}{=} \frac{[\det(\mathcal{F})]}{\mathrm{rk}(\mathcal{F})} \equiv \frac{\ell^{2g-1}[\mathcal{L}]}{\ell^{2g}} \equiv \frac{[\mathcal{L}]}{\ell}. \qquad \square$$

1.3.8. Proposition. — *Fix some $\mu \in \mathrm{NS}(A)\otimes\mathbb{Q}$.*

(i) *There exists a simple semi-homogeneous vector bundle $\mathcal{E}$ on A of slope μ.*

(ii) *Every other such simple semi-homogeneous vector bundle $\mathcal{E}'$ of slope μ is of the form $\mathcal{E}' \simeq \mathcal{E}\otimes\mathcal{M}$ for some line bundle $\mathcal{M}\in\mathrm{Pic}^0(A)$.*

(iii) *The rank of $\mathcal{E}$ can be computed using $\mathrm{rk}(\mathcal{E})^2 = \deg(\mathrm{pr}_1|_{\Phi_\mu})$. Also $\chi(\mathcal{E})^2 = \deg(\mathrm{pr}_2|_{\Phi_\mu})$, if $\chi(\mathcal{E})\neq 0$.*[(9)]

Proof. — See [Muk78, Thm. 7.11]. Part (i) uses the constructions recalled in Proposition 1.3.7. $\square$

1.3.9. Remark. — By ¶1.3.6 and Proposition 1.3.8.(iii), the fact that $\#A[\ell] \leq \ell^{2g}$ implies that the rank of a simple semi-homogeneous vector bundle $\mathcal{E}$ of slope $\mu = [\mathcal{L}]\otimes\frac{1}{\ell}$ satisfies the bound $\mathrm{rk}(\mathcal{E}) \leq \ell^g$. On the other hand $\deg([\ell]) = \ell^{2g}$, so $[\ell]_*\mathcal{L}^{\otimes\ell}$ has rank ℓ^{2g} and is *not* simple.

1.3.10. — Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two simple semi-homogeneous vector bundles of the same slope $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$. Then by [Muk78, Prop. 6.17] and [Orl02, Lem. 4.8] we have either $\mathcal{E}_1 \simeq \mathcal{E}_2$ or

$$\mathrm{Ext}^\bullet(\mathcal{E}_1,\mathcal{E}_2) = 0 \quad \text{and} \quad \mathrm{Ext}^\bullet(\mathcal{E}_2,\mathcal{E}_1) = 0.$$

In the case $\mathcal{E}_1 \simeq \mathcal{E}_2 =: \mathcal{E}$, we have for $j = 1,\dots,g$

$$\dim \mathrm{Ext}^j(\mathcal{E},\mathcal{E}) = \binom{g}{j}$$

by [Muk78, Thm. 5.8] and the local to global Ext spectral sequence.

(9) By ¶1.3.6, $\Sigma_\mu = \ker(\mathrm{pr}_1\colon \Phi_\mu \to A)$ is finite, so $\mathrm{pr}_1|_{\Phi_\mu}$ is indeed an isogeny. Note that $\mathrm{pr}_2|_{\Phi_\mu}$ is an isogeny if and only if $\varphi_{\det(\mathcal{E})}\colon A \to A^\vee$ is an isogeny. But [Muk78, Prop. 6.12] says that $\chi(\det(\mathcal{E})) = \mathrm{rk}(\mathcal{E})^{g-1}\chi(\mathcal{E}) \neq 0$, which implies that $\varphi_{\det(\mathcal{E})}$ is an isogeny by the Riemann–Roch theorem, cf. [EGM, Thm. 9.11].

***1.3.11. Definition* (Unipotent vector bundles).** — A vector bundle $\mathcal{U}$ on a variety X is called *unipotent* if it admits an increasing filtration $0 = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_r = \mathcal{U}$ such that $\mathcal{U}_i/\mathcal{U}_{i-1} \simeq \mathcal{O}_X$ for every $i = 1, \dots, r$.

A particular class of homogeneous vector bundles on abelian varieties are the unipotent vector bundles, and conversely every homogeneous vector bundle $\mathcal{E}$ on A is the direct sum of unipotent vector bundles twisted by algebraically trivial line bundles, cf. [Muk78, Thm. 4.17], that is, $\mathcal{E} \simeq \bigoplus_i \mathcal{U}_i \otimes \mathcal{L}_i$ for some unipotent vector bundles $\mathcal{U}_i$ on A and line bundles $\mathcal{L}_i \in \mathrm{Pic}^0(A)$.

1.3.12. Proposition. — *Let $\mathcal{U}$ be a unipotent vector bundle on a variety X with $\mathrm{H}^0(X, \mathcal{O}_X) = \Bbbk$, and set $r = \mathrm{rk}(\mathcal{U})$, then*

(i) *$\dim \mathrm{End}(\mathcal{U}) \le r^2$, and*

(ii) *$\dim \mathrm{End}(\mathcal{U}) = r^2$ if and only if $\mathcal{U}$ is split, i.e. $\mathcal{U} \simeq \mathcal{O}_X^{\oplus r}$.*

Proof. — We use induction on the rank r. First note that indeed $\mathrm{rk}(\mathcal{U}_i) = i$, so in the base case $i = 1$ we just have $\mathcal{U} = \mathcal{U}_1 \simeq \mathcal{O}_X$. Now consider the induction step and abbreviate $\mathcal{U}' := \mathcal{U}_{r-1}$ and $\mathcal{U} := \mathcal{U}_r$, as well as $\mathcal{O} := \mathcal{O}_X$. Then we have a short exact sequence $0 \to \mathcal{U}' \to \mathcal{U} \to \mathcal{O} \to 0$ and applying long exact Ext sequences, we obtain the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & \mathrm{Ext}^1(\mathcal{O},\mathcal{U}') & \mathrm{Ext}^1(\mathcal{O},\mathcal{U}) & \mathrm{Ext}^1(\mathcal{O},\mathcal{O}) & \\
0 \arrow[r] & \mathrm{Hom}(\mathcal{U}',\mathcal{U}') \arrow[r] \arrow[u] & \mathrm{Hom}(\mathcal{U}',\mathcal{U}) \arrow[r] \arrow[u] & \mathrm{Hom}(\mathcal{U}',\mathcal{O}) \arrow[r] \arrow[u] & \mathrm{Ext}^1(\mathcal{U}',\mathcal{U}') \\
0 \arrow[r] & \mathrm{Hom}(\mathcal{U},\mathcal{U}') \arrow[r] \arrow[u] & \mathrm{Hom}(\mathcal{U},\mathcal{U}) \arrow[r] \arrow[u] & \mathrm{Hom}(\mathcal{U},\mathcal{O}) \arrow[r] \arrow[u] & \mathrm{Ext}^1(\mathcal{U},\mathcal{U}') \\
0 \arrow[r] & \mathrm{Hom}(\mathcal{O},\mathcal{U}') \arrow[r] \arrow[u] & \mathrm{Hom}(\mathcal{O},\mathcal{U}) \arrow[r] \arrow[u] & \mathrm{Hom}(\mathcal{O},\mathcal{O}) \arrow[r, "\delta"] \arrow[u] & \mathrm{Ext}^1(\mathcal{O},\mathcal{U}') \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

By assumption we have $\mathrm{Hom}(\mathcal{O}, \mathcal{O}) = \Bbbk$, so the third column tells us that $\dim \mathrm{Hom}(\mathcal{U}, \mathcal{O}) \le \dim \mathrm{Hom}(\mathcal{U}', \mathcal{O}) + 1$, and inductively we get $\dim \mathrm{Hom}(\mathcal{U}_i, \mathcal{O}) \le i$. Looking at the bottom row, we similarly get

$$\dim \mathrm{Hom}(\mathcal{O}, \mathcal{U}_i) \le i,$$

and note that equality holds if and only if the unipotent filtration splits (in every induction step) since the connecting map δ must be zero in this case. Then the third row and first column, respectively, imply that

$$\dim \mathrm{Hom}(\mathcal{U}, \mathcal{U}) \le \dim \mathrm{Hom}(\mathcal{U}, \mathcal{U}') + r$$
$$\dim \mathrm{Hom}(\mathcal{U}, \mathcal{U}') \le \dim \mathrm{Hom}(\mathcal{U}', \mathcal{U}') + (r - 1).$$

Combining all these inequalities yields

$$\dim \operatorname{Hom}(\mathcal{U},\mathcal{U}) \leq \dim \operatorname{Hom}(\mathcal{U}',\mathcal{U}') + 2r-1 \leq (r-1)^2 + 2r - 1 = r^2.$$

Finally $\dim \operatorname{End}(\mathcal{U}) = r^2$ forces all these inequalities above to become equalities, which implies that $\mathcal{U}$ is split, as remarked before. □

1.3.13. — Fix a vector bundle $\mathcal{E}$. Generalizing the notion of unipotent vector bundles, a vector bundle $\mathcal{F}$ on A is called *$\mathcal{E}$-potent* if $\mathcal{F}$ admits a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_r = \mathcal{F}$ such that $\mathcal{F}_i/\mathcal{F}_{i-1} \simeq \mathcal{E}$. Denote the category of $\mathcal{E}$-potent vector bundles on A by $\mathbf{Bun}_{\mathcal{E}-\mathrm{potent}}(A)$ and the category of unipotent vector bundles on A by $\mathbf{Bun}_{\mathrm{unip}}(A)$. We have a natural functor

$$\mathbf{Bun}_{\mathrm{unip}}(A) \to \mathbf{Bun}_{\mathcal{E}-\mathrm{potent}}(A) \quad \mathcal{U} \mapsto \mathcal{U} \otimes \mathcal{E},$$

which is an equivalence of categories by [Muk78, Prop. 6.2] if $\mathcal{E}$ is a simple semi-homogeneous vector bundle and $\mathrm{char}(\Bbbk) \nmid \mathrm{rk}(\mathcal{E})$. In particular, under these latter assumptions, for two unipotent vector bundles $\mathcal{U}_1$, $\mathcal{U}_2$ we have

$$\mathcal{U}_1 \otimes \mathcal{E} \simeq \mathcal{U}_2 \otimes \mathcal{E} \quad \text{if and only if} \quad \mathcal{U}_1 \simeq \mathcal{U}_2.$$

1.3.14. — Let $\mathcal{F}$ be a semi-homogeneous vector bundle of slope $\mu(\mathcal{F}) = \mu$. By Proposition 1.3.7 there exists a Jordan–Hölder filtration $\mathcal{F}_\bullet \subset \mathcal{F}$, whose graded pieces $\mathcal{E}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ are simple semi-homogeneous vector bundles of slope $\mu(\mathcal{E}_i) = \mu$. Using the Ext-orthongonality from ¶1.3.10 for the bundles $\mathcal{E}_i$, i.e. either

$$\mathcal{E}_i \simeq \mathcal{E}_{i'} \quad \text{or} \quad \operatorname{Ext}^\bullet(\mathcal{E}_i, \mathcal{E}_{i'}) = 0,$$

we can rearrange the filtration $\mathcal{F}_\bullet$ into the form

$$\mathcal{F} \simeq \bigoplus_{j \in J} \mathcal{U}_j \otimes \mathcal{E}_j,$$

where the $\mathcal{U}_j$ are unipotent vector bundles and J is a subset of the indices i of the filtration $\mathcal{F}_\bullet$ such that the vector bundles $\mathcal{E}_j$ are pairwise distinct, cf. [Muk78, Prop. 6.18].

1.3.15. Remark. — As a teaser for the study of derived equivalences later on, let us discuss the interaction of Mukai's original Fourier–Mukai transform with unipotent and homogeneous bundles. Following [Muk78, §4], consider the functor

$$\Phi\colon \mathbf{D}^{\mathrm{b}}(A^\vee) \to \mathbf{D}^{\mathrm{b}}(A) \quad \mathcal{F} \to \mathbf{R}\mathrm{pr}_{A,*}(\mathrm{pr}_{A^\vee}^*(\mathcal{F}) \otimes \mathcal{P}),$$

between bounded derived categories of coherent sheaves (see §2.1 for a recollection), where $\mathcal{P} \in \operatorname{Pic}(A \times A^\vee)$ is the Poincaré bundle and pr_A and $\mathrm{pr}_{A^\vee}$ denote the projections onto the respective factors. By [Muk81, Thm. 2.2], this functor is an equivalence of categories. Since $\mathrm{t}_{(a,0)}^*\mathcal{P} \simeq \mathcal{P} \otimes \mathcal{P}_a$, we have by [Muk78, Lem. 4.3]

$$\mathrm{t}_a^*\Phi(\mathcal{F}) \simeq \Phi(\mathcal{F} \otimes \mathcal{P}_a). \tag{1.3.1}$$

Denote by $\mathbf{Coh}_{\{0\}}(A^\vee)$ the category of coherent sheaves supported at the origin, by $\mathbf{Coh}_{\mathrm{fin}}(A^\vee)$ the category of coherent sheaves supported at finitely many points, and by $\mathbf{Bun}_{\mathrm{homog}}(A)$ the category of homogeneous vector bundles. Then [Muk78, Lem. 4.8] tells us, using (1.3.1), that Φ restricts to functors

$$\Phi\colon \mathbf{Coh}_{\{0\}}(A^\vee) \to \mathbf{Bun}_{\mathrm{unip}}(A),$$
$$\Phi\colon \mathbf{Coh}_{\mathrm{fin}}(A^\vee) \to \mathbf{Bun}_{\mathrm{homog}}(A),$$

which are equivalences by [Muk78, Thm. 4.12, Thm. 4.19], and satisfy $\mathrm{rk}(\Phi(\mathcal{F})) = \mathrm{length}(\mathcal{F})$.

Taking the discussion one step further, by taking formal completions at the origin, we have an equivalence $\mathbf{Coh}_{\{0\}}(A^\vee) \simeq \mathbf{Coh}_{\{0\}}(\mathrm{Spf}(\Bbbk[\![x_1, \ldots, x_g]\!]))$, and the latter can be described as the category of finite-dimensional $\Bbbk$-vector spaces V together with g commuting nilpotent endomorphism $\psi_i\colon V \to V$. This gives also an explanation of Proposition 1.3.12 using linear algebra, since a nilpotent endomorphism of V which commutes with every endomorphism in $\mathrm{End}(V)$ must be trivial.

CHAPTER 2

Fourier–Mukai equivalences

2.1. Derived categories and their (auto)equivalences

See the textbook [HuyFM] for a general introduction to derived categories of coherent sheaves and Fourier–Mukai functors. This section is completely expository.

***2.1.1.* Derived categories of coherent sheaves.** — Let X be a variety over a field $\Bbbk$, then the category $\mathbf{Coh}(X)$ of coherent sheaves on X is a $\Bbbk$-linear abelian category. Consider the category $\mathbf{Kom}(\mathbf{Coh}(X))$ of complexes of coherent sheaves and the class of so called *quasi-isomorphisms*, consisting of morphisms $f\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ which become an isomorphism after taking cohomology of complexes. Localizing at the class of quasi-isomorphisms, i.e. essentially adjoining inverses of quasi-isomorphisms, gives rise to the derived category

$$Q\colon\ \mathbf{Kom}(\mathbf{Coh}(X)) \to \mathbf{D}(\mathbf{Coh}(X)),$$

which can be constructed such that Q is the identity on objects, cf. [HuyFM, Thm. 2.10, Cor. 2.11]. Similarly, by considering bounded complexes, we obtain the *bounded derived category of coherent sheaves*

$$\mathbf{D}^{\mathrm{b}}(X) := \mathbf{D}^{\mathrm{b}}(\mathbf{Coh}(X)).$$

The (bounded) derived category is still a $\Bbbk$-linear category which becomes a *triangulated category* by considering the autoequivalence

$$[1]\colon\ \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X)$$

which shifts the indices of a complex by $+1$, and declaring the mapping cone sequences

$$\mathcal{F}^\bullet \xrightarrow{f} \mathcal{G}^\bullet \to \mathrm{cone}(f) \to \mathcal{F}^\bullet[1]$$

as the *distinguished triangles*, cf. [HuyFM, Prop. 2.24].

With a view towards derived functors, recall that a $\Bbbk$-linear functor $\Phi\colon \mathscr{T} \to \mathscr{T}'$ between triangulated categories is *exact* if it commutes with the respective shift functors, i.e. $\Phi \circ [1] \simeq [1] \circ \Phi$, and maps distinguished triangles in $\mathscr{T}$ to distinguished triangles in $\mathscr{T}'$.

2.1.2. — Since the category of coherent sheaves almost never has enough injective objects, one needs to consider for the construction of derived functors the category $\mathbf{QCoh}(X)$ of quasi-coherent sheaves on X, which has enough injective objects. On the level of derived categories, the natural functor $\mathbf{D}^{\mathrm{b}}(\mathbf{Coh}(X)) \to \mathbf{D}^{\mathrm{b}}(\mathbf{QCoh}(X))$ provides an equivalence

$$\mathbf{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbf{QCoh}(X)) \tag{2.1.1}$$

onto the full subcategory of complexes of quasi-coherent sheaves which have coherent cohomology sheaves, cf. [HuyFM, Prop. 3.5].

Let $f\colon X \to Y$ be a morphism of varieties, then the direct image functor $f_*\colon \mathbf{QCoh}(X) \to \mathbf{QCoh}(Y)$ is left exact. By considering an injective resolution of a given complex and applying f_* to such a resolution component-wise, we obtain the derived functor

$$\mathbf{R}f_*\colon \mathbf{D}^{\mathrm{b}}(\mathbf{QCoh}(X)) \to \mathbf{D}^{\mathrm{b}}(\mathbf{QCoh}(Y)), \tag{2.1.2}$$

which is well-defined, $\Bbbk$-linear, and exact, cf. [HuyFM, Prop. 2.47, Thm. 3.22]. If the morphism f is proper, then the higher direct image sheaves $\mathrm{H}^i(\mathbf{R}f_*\mathcal{F})$ of a coherent sheaf $\mathcal{F} \in \mathbf{Coh}(X)$ are itself coherent and $\mathrm{H}^i(\mathbf{R}f_*\mathcal{F}) = 0$ for $|i| \gg 0$, cf. [HuyFM, Thm. 3.23], so using (2.1.1) we can restrict (2.1.2) to a functor

$$\mathbf{R}f_*\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y).$$

Let $\mathcal{F} \in \mathbf{Coh}(X)$ be a coherent sheaf. Then the tensor product functor

$$\mathcal{F} \otimes (-)\colon \mathbf{Coh}(X) \to \mathbf{Coh}(X)$$

is right exact, and restricted to the subcategory of locally free sheaves, it is exact. When X is projective, the category of coherent sheaves on X has enough locally free sheaves, cf. [HuyFM, Prop. 3.18]. Assume that X is smooth, then any coherent sheaf $\mathcal{F} \in \mathbf{Coh}(X)$ admits a resolution by locally free sheaves which is of finite length (bounded by the dimension of X, in fact), cf. [HuyFM, Prop. 3.26]. So tensoring with a locally free resolution yields the left derived exact functor

$$\mathcal{F} \overset{\mathbf{L}}{\otimes} (-)\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X).$$

Following [HuyFM, pp. 79–80], this generalizes to the case where $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ is a complex of coherent sheaves and leads to the bifunctor

$$(-) \overset{\mathbf{L}}{\otimes} (-)\colon \mathbf{D}^{\mathrm{b}}(X) \times \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X).$$

Let $f\colon X \to Y$ be a morphism of smooth varieties. One essentially defines the derived pullback

$$\mathbf{L}f^*\colon \mathbf{D}^{\mathrm{b}}(Y) \to \mathbf{D}^{\mathrm{b}}(X)$$

as the composition of the exact inverse image functor with the derived tensor product, cf. [HuyFM, p. 81].

2.1.3. Remark. — The functors discussed in ¶2.1.2 satisfy the expected compatibilities. For example, in telegraphic style:

– $\mathbf{R}f_* \circ \mathbf{R}g_* = \mathbf{R}(f \circ g)_*$
– $\mathbf{L}f^* \circ \mathbf{L}g^* = \mathbf{L}(g \circ f)^*$
– $\mathbf{R}f_* \mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G} \simeq \mathbf{R}f_*(\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathcal{G})$ (projection formula)
– $(\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G}) \overset{\mathbf{L}}{\otimes} \mathcal{H} \simeq \mathcal{F} \overset{\mathbf{L}}{\otimes} (\mathcal{G} \overset{\mathbf{L}}{\otimes} \mathcal{H})$
– $\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G} \simeq \mathcal{G} \overset{\mathbf{L}}{\otimes} \mathcal{F}$
– $\mathbf{L}f^*(\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G}) \simeq \mathbf{L}f^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathcal{G}$
– etc.

2.1.4. Situation. — Let X and Y be smooth, projective varieties over a field $\Bbbk$. The product $X \times Y$ denotes the product in the category of varieties over $\Bbbk$, i.e. the fiber product $X \times_{\Bbbk} Y$.

***2.1.5. Definition* (Derived equivalences).** —
(i) A functor $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ is a *derived equivalence* if it is $\Bbbk$-linear, an equivalence of categories, and exact in the sense of triangulated categories.
(ii) The *Fourier-Mukai functor* $\mathrm{FM}_{\mathcal{P}}\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ associated to a *kernel* $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(X \times Y)$ is defined as

$$\mathrm{FM}_{\mathcal{P}}(-) \coloneqq \mathbf{R}\mathrm{pr}_{Y,*}(\mathbf{L}\mathrm{pr}_X^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{P}),$$

where $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ denote the coordinate projections.

We say that X and Y are *derived equivalent*, or *Fourier–Mukai partners*, if there exists some derived equivalence between $\mathbf{D}^{\mathrm{b}}(X)$ and $\mathbf{D}^{\mathrm{b}}(Y)$.

2.1.6. Notation. — Denote by $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))$ the group of isomorphism classes of derived autoequivalences of X, and denote by $\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))$ the $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))$-torsor of isomorphism classes of derived equivalences between X and Y.

2.1.7. Remark. — We do not need to derive the pullback functor pr_X^* in Definition 2.1.5, since the projection pr_X is flat. Similarly, if $\mathcal{P}$ is a complex of flat coherent sheaves on $X \times Y$, e.g. $\mathcal{P}$ is a locally free sheaf, then the derived tensor product in Definition 2.1.5 becomes a usual tensor product.

***2.1.8.* Composition and convolution.** — Of course one can consider the composition $\Phi' \circ \Phi$ of two functors $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ and $\Phi'\colon \mathbf{D}^{\mathrm{b}}(Y) \to \mathbf{D}^{\mathrm{b}}(X)$. Now assume that Φ and Φ' are Fourier–Mukai functors with kernel $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(X \times Y)$ and $\mathcal{Q} \in \mathbf{D}^{\mathrm{b}}(Y \times Z)$, respectively, and define the convolution product of $\mathcal{Q}$ and $\mathcal{P}$ as

$$\mathcal{Q} \star \mathcal{P} \coloneqq \mathbf{R}\mathrm{pr}_{XZ,*}(\mathbf{L}\mathrm{pr}_{XY}^*\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathbf{L}\mathrm{pr}_{YZ}^*\mathcal{Q}),$$

where pr_{XZ}, pr_{XY} and pr_{YZ} are the projections from the product $X \times Y \times Z$ that are suggested by their name. Then Mukai explained, cf. [HuyFM, Prop. 5.10], that composition of functors corresponds to convolution of kernels, i.e.

$$\mathrm{FM}_{\mathcal{Q}} \circ \mathrm{FM}_{\mathcal{P}} \simeq \mathrm{FM}_{\mathcal{Q}\star\mathcal{P}}\,.$$

Note that the convolution product is functorial in both entries.

2.1.9. Example. — The variety X admits the following *standard autoequivalences*.
(i) The shift functor $[n]\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X)$ which shifts the indices of a complex by n is an autoequivalence for every $n \in \mathbb{Z}$.
(ii) Let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle, then twisting complexes by $\mathcal{L}$ gives an autoequivalence $- \otimes \mathcal{L} \in \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))$.

(iii) Let $f \in \mathrm{Aut}_{\Bbbk}(X)$ be an automorphism of the variety $X/\Bbbk$, then push-forward along f provides an autoequivalence $f_* \in \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))$. Equivalently, pullback along f provides an autoequivalence, but we have the identity of functors

$$f^* = (f^{-1})_* \colon \mathbf{QCoh}(X) \to \mathbf{QCoh}(X), \tag{2.1.3}$$

as can be verified directly or by an application of affine base change, cf. [SP, Tag 02KG].

Note that the described functors, on the level of coherent sheaves, are exact, so one does not need to derive them in order to view them as functors between derived categories. Letting $\mathrm{Aut}_{\Bbbk}(X)$ act on $\mathrm{Pic}(X)$ via $f.\mathcal{L} := f_*\mathcal{L}$, we have in conclusion the homomorphism

$$\begin{aligned} \mathbb{Z} \times \mathrm{Aut}_{\Bbbk}(X) \ltimes \mathrm{Pic}(X) &\hookrightarrow \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X)) \\ (n, f, \mathcal{L}) &\mapsto \mathcal{L}[n] \otimes f_*(-), \end{aligned} \tag{2.1.4}$$

which is seen to be injective, if X is geometrically reduced, by considering the trivial sheaf $\mathcal{O}_X$ as well as skyscraper sheaves $\mathrm{k}(x)$ for closed points $x \in X$.

***2.1.10. Theorem* (Orlov).** — *Let X and Y be smooth, projective varieties over a field $\Bbbk$, and let $\Phi \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be an exact $\Bbbk$-linear fully faithful functor. Then there exists a kernel $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(X \times_{\Bbbk} Y)$ and a natural isomorphism*

$$\Phi \simeq \mathrm{FM}_{\mathcal{P}} \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y).$$

Furthermore, the kernel $\mathcal{P}$ is unique up to isomorphism.

Proof. — See [Orl97, Thm. 2.2]. The additional assumptions about the existence of adjoints in loc. cit. are taken care of using [BB03]. See [Bal09; Bal11] for generalizations to the non-smooth case. ☐

2.1.11. Example. — Let us describe the Fourier–Mukai kernels of the equivalences in Example 2.1.9; see [HuyFM, Ex. 5.4] for details.

(i) Let $f \colon X \to Y$ be a morphism of varieties, and denote by $\imath \colon \Gamma_f \hookrightarrow X \times Y$ the graph of f. Then the push-forward functor is described as a Fourier–Mukai functor as

$$\mathbf{R}f_* \simeq \mathrm{FM}_{\mathcal{O}_{\Gamma_f}} \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y).$$

(ii) Consider the transpose $\Gamma_f^{\mathrm{t}} \hookrightarrow Y \times X$ of the graph of f, which is $\imath$ composed with the isomorphism swapping the factors of the product $X \times Y$. Then the pullback functor is described as

$$\mathbf{L}f^* \simeq \mathrm{FM}_{\mathcal{O}_{\Gamma_f^{\mathrm{t}}}} \colon \mathbf{D}^{\mathrm{b}}(Y) \to \mathbf{D}^{\mathrm{b}}(X).$$

(iii) Let $\mathcal{L} \in \mathrm{Pic}(X)$ and set $f = \mathrm{id}$, so $\imath \colon \Delta = \Gamma_{\mathrm{id}} \hookrightarrow X \times X$ is the diagonal. Then the twist functor is described as

$$- \otimes \mathcal{L} \simeq \mathrm{FM}_{\imath_*\mathcal{L}} \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X).$$

(iv) For $n \in \mathbb{Z}$ we have

$$[n] = \mathrm{FM}_{\mathcal{O}_\Delta[n]} \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X).$$

In view of the Poincaré bundle $\mathcal{P} \in \mathrm{Pic}(A^\vee \times A)$ for an abelian variety A, cf. ¶1.2.8, the following example is of interest, cf. [HuyFM, Ex. 5.4.vi].

(v) Let $\mathcal{P} \in \mathbf{Coh}(X \times Y)$ be flat over X, and let $x \in X(\Bbbk)$ be a rational point, then

$$\mathrm{FM}_{\mathcal{P}}(\mathrm{k}(x)) \simeq \mathcal{P}|_{\{x\}\times Y}.$$

The following proposition describing the the Fourier–Mukai functor of a pulled-back kernel will be useful when studying derived autoequivalences of a variety with group action in §5.2.

2.1.12. Proposition. — *Let $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(X \times Y)$, and let $g\colon X' \to X$ and $h\colon Y' \to Y$ be morphisms of varieties. Then we have the identity*

$$\mathrm{FM}_{\mathbf{L}(g,h)^*\mathcal{P}} \simeq \mathbf{L}h^* \circ \mathrm{FM}_{\mathcal{P}} \circ \mathbf{R}g_*.$$

Proof. — See [HuyFM, Exer. 5.12]. This is a straight-forward calculation with Fourier–Mukai kernels; some details are given in [Plo05, Ex. 1.6.(4)]. ☐

2.1.13. Proposition. — *Let $\mathrm{FM}_{\mathcal{P}}\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a derived equivalence with Fourier–Mukai kernel $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(X \times Y)$. Assume that the variety X satisfies* $\mathrm{H}^0(X, \mathcal{O}_X) = \Bbbk$.(1) *Then $\mathcal{P}$ is a* simple object, *i.e.*

$$\mathrm{Hom}(\mathcal{P}, \mathcal{P}) = \Bbbk.$$

More generally, convolution with $\mathcal{P}$ induces an isomorphism

$$\mathcal{P}\star\colon\ \mathrm{Ext}^\bullet(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) \xrightarrow{\sim} \mathrm{Ext}^\bullet(\mathcal{P}, \mathcal{P}),$$

where the left hand side is called the Hochschild cohomology *of X.*

Proof. — See for example [Plo05, Lem. 1.12] for a proof. The argument uses that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X\times X)}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) &\simeq \mathrm{Hom}_{\mathbf{Coh}(X\times X)}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})\\ &\simeq \mathrm{Hom}(\Delta_{X,*}\mathcal{O}_X, \Delta_{X,*}\mathcal{O}_X) \simeq \mathrm{Hom}(\Delta_X^*\Delta_{X,*}\mathcal{O}_X, \mathcal{O}_X) \simeq \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) \simeq \Bbbk,\end{aligned}$$

by, among other reasons, adjunction and the fact that $\Delta_X^*\Delta_{X,*}\mathcal{O}_X \simeq \mathcal{O}_X$, since $\Delta_X\colon X \to X \times X$ is a closed immersion.

For an argument for the more general claim, see [AT14, §6.1]. ☐

2.1.14. — The notion of derived equivalence does not always provide more flexibility in comparison to the notion of isomorphism. For example, by [BO01, Thm. 2.5, Thm. 3.1], if X is an (anti-)*Fano variety*, i.e. X is a connected, smooth, projective variety with (anti-)ample anticanonical bundle $\omega_X^\vee$, then for any smooth variety X'

$$\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(X') \quad \text{implies} \quad X \simeq X',$$

and the group of autoequivalences consists of standard autoequivalences

$$\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X)) \simeq \mathbb{Z} \times \mathrm{Aut}_{\Bbbk}(X) \ltimes \mathrm{Pic}(X).$$

(1) For example, assume that X, in addition to being smooth and proper over $\Bbbk$, is geometrically connected, e.g. it is connected with a $\Bbbk$-rational point, cf. [LiuAG, Cor. 3.3.21], [SP, Tag 04KV].

One might use this and the study of derived equivalences of minimal surfaces in [BM01] as a reason to focus on the class of varieties with (numerically) trivial canonical bundle. But *Enriques surfaces* and *bielliptic surfaces* have still no non-trivial Fourier–Mukai partners, over an algebraically closed field $\overline{\mathbb{k}}$ with $\mathrm{char}(\overline{\mathbb{k}}) \neq 3, 5$, cf. [BM01, Prop. 6.1, Prop. 6.2], [HLT21, Thm 1.1, Thm. 1.2].

The case of K3 surfaces is in general richer, see [HuyK3, Rmk. 2.10] for an overview; but again, two K3 surfaces S and S' over $\overline{\mathbb{k}}$ which are derived equivalent must be isomorphic as soon as the Picard rank of S satisfies $\rho(S) > 11$, see [Muk87, Prop. 6.2] in view of [BM01, Thm. 5.1], and [LO15, Thm. 1.1]. In particular, two Kummer K3 surfaces, cf. Example 1.1.6, are derived equivalent if and only if they are isomorphic. Nevertheless, the group $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(S))$ can still be interesting.

2.2. Equivariant derived categories and Ploog's method

The general theory of derived categories of equivariant sheaves is developed in [BL], where the case of discrete groups is spelled out in Section 8. See [BBH, §1.4] or [BKR01, §4] for a summary of the construction of Fourier–Mukai functors in the equivariant setting, and [Blu07, Ch. 3] for some details about equivariant sheaves.

In particular, we state a theorem (Theorem 2.2.13) originally due to Ploog in mildly greater generality, contribute a technical proposition about the linearization-obstruction of certain *non-simple* sheaves, and check explicitly the assumptions required to apply the derived McKay correspondence to generalized Kummer varieties.

2.2.1. Situation. — Let G, respectively H, be a finite group, acting on an algebraic variety X, respectively Y, over a field $\Bbbk$. We will make further assumptions on the characteristic of $\Bbbk$ in Situation 2.2.6 below.

***2.2.2. Definition* (Equivariant sheaves).** — A sheaf $\mathcal{F} \in \mathbf{Mod}(\mathcal{O}_X)$ on X is called *G-invariant* if there exist isomorphisms $\mathcal{F} \simeq g^*\mathcal{F}$ for each $g \in G$. A *G-equivariant structure* λ (also called *G-linearization*) on $\mathcal{F}$ is given by isomorphisms

$$\lambda_g \colon \mathcal{F} \xrightarrow{\sim} g^*\mathcal{F}$$

for each $g \in G$, subject to the cocycle condition that $\lambda_1 = \mathrm{id}_{\mathcal{F}}$ and

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\lambda_{gh}"] \arrow[d, "\lambda_h"'] & (gh)^*\mathcal{F} \arrow[d, "\simeq"] \\ h^*\mathcal{F} \arrow[r, "h^*\lambda_g"] & h^*g^*\mathcal{F} \end{tikzcd}$$

commutes for $g, h \in G$. A *G-equivariant sheaf* $(\mathcal{F}, \lambda)$ is a sheaf $\mathcal{F} \in \mathbf{Mod}(\mathcal{O}_X)$ together with a G-equivariant structure λ on $\mathcal{F}$.

2.2.3. Remark. — More generally, let $\mathscr{C}$ be a category endowed with a categorical action of G, cf. [Del97; Sos12]. The notion of equivariant sheaf from Definition 2.2.2 generalizes directly to the notion of *equivariant object*, and we denote the category having the latter as objects by

$$\mathscr{C}^{\mathrm{h}G},$$

where the symbol "h" in the notation is motivated by the concept of *homotopy fixed points.*

2.2.4. Example. — Let us describe a couple of equivariant sheaves that are available in general.

(i) The structure sheaf $\mathcal{O}_X$ carries a canonical G-equivariant structure coming from the canonical isomorphsim $g^*\mathcal{O}_X \simeq \mathcal{O}_X$.

(ii) The derivative maps $\mathrm{d}g \colon g^*\,\Omega^1_X \to \Omega^1_X$ for $g \in G$ satisfy $\mathrm{d}(gh) = \mathrm{d}g \circ h^*\mathrm{d}g$ by their construction, so the inverses

$$\lambda_g^{\Omega^1_X} := (\mathrm{d}g)^{-1} = g^*\mathrm{d}(g^{-1})$$

endow the cotangent sheaf Ω^1_X with an equivariant structure.

(iii) Assume that X is smooth, then the canonical sheaf ω_X is given by $\det \Omega^1_X$, and it inherits an equivariant structure from Ω^1_X by taking determinants,

$$\lambda_g^{\omega_X} := \det(\lambda_g^{\Omega^1_X}).$$

(iv) Let $\chi\colon G \to \Bbbk^\times$ be a character of the group G, and $(\mathcal{F}, \lambda)$ an equivariant sheaf on X. Then

$$\lambda_g^\chi := \chi(g) \cdot \lambda_g$$

provides another equivariant sheaf $(\mathcal{F}, \lambda^\chi)$. If $\mathcal{F}$ is in addition simple, meaning $\mathrm{End}_{\mathcal{O}_X}(\mathcal{F}) = \Bbbk$, then the set of equivariant structures on $\mathcal{F}$ is a torsor under the *character group* $\widehat{G} = \mathrm{Hom}(G, \Bbbk^\times)$, cf. [Plo05, Lem. 3.5.(ii)].

Consider the n-fold product $X = X_0^n$ of a variety X_0 endowed with the permutation action by $G = \mathrm{S}_n$, and let $\mathcal{F} \in \mathbf{Mod}(\mathcal{O}_{X_0})$ be some sheaf on X_0.

(v) The box product $\mathcal{F}^{\boxtimes n} := \mathrm{pr}_1^*\mathcal{F} \otimes \cdots \otimes \mathrm{pr}_n^*\mathcal{F}$ carries a canonical equivariant structure

$$\lambda_\sigma^{\mathcal{F}^{\boxtimes n}} \colon \mathcal{F}^{\boxtimes n} \to \sigma^*\mathcal{F}^{\boxtimes n} \simeq \mathrm{pr}^*_{\sigma^{-1}(1)}\mathcal{F} \otimes \cdots \otimes \mathrm{pr}^*_{\sigma^{-1}(n)}\mathcal{F}$$

given by the braiding isomorphisms of the tensor product.

2.2.5. — Common constructions for sheaves of $\mathcal{O}_X$-modules make sense for equivariant sheaves. Let $(\mathcal{F}, \lambda^{\mathcal{F}})$ and $(\mathcal{G}, \lambda^{\mathcal{G}})$ be G-equivariant sheaves on X.

(i) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ inherits the equivariant structure

$$\lambda_g^{\mathcal{H}om(\mathcal{F},\mathcal{G})}(\psi) := \lambda_g^{\mathcal{G}} \circ g^*(\psi) \circ (\lambda_g^{\mathcal{F}})^{-1}.$$

The fixed points of the induces right action on global sections are exactly the homomorphism $\mathcal{F} \to \mathcal{G}$ which are compatible with $\lambda^{\mathcal{F}}$ and $\lambda^{\mathcal{G}}$.

(ii) $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ inherits the equivariant structure

$$\lambda_g^{\mathcal{F}\otimes\mathcal{G}} := \lambda_g^{\mathcal{F}} \otimes \lambda_g^{\mathcal{G}} \colon \mathcal{F} \otimes \mathcal{G} \to g^*\mathcal{F} \otimes g^*\mathcal{G} \simeq g^*(\mathcal{F} \otimes \mathcal{G}).$$

Let $f\colon X \to Y$ be an equivariant morphism, relative to some given homomorphism $\varphi\colon G \to H$ which we assume to be surjective in (iv), and let $(\mathcal{H}, \lambda^{\mathcal{H}})$ be an H-equivariant sheaf on Y.

(iii) $f^*\mathcal{H}$ inherits the equivariant structure

$$\lambda_g^{f^*\mathcal{H}} := f^*\lambda^{\mathcal{H}}_{\varphi(g)} \colon f^*\mathcal{H} \to f^*\varphi(g)^*\mathcal{H} \simeq g^*f^*\mathcal{H},$$

where the last isomorphism comes from the equivariance condition $\varphi(g) \circ f = f \circ g$.

(iv) $f_*\mathcal{F}$ inherits the G-equivariant structure

$$\lambda_g^{f_*\mathcal{F}} := f_*\lambda_g^{\mathcal{F}} \colon f_*\mathcal{F} \to f_*g^*\mathcal{F} \simeq \varphi(g)^*f_*\mathcal{F},$$

where the last isomorphism comes from affine base change, cf. [SP, Tag 02KG]. Since $\ker(\varphi)$ acts trivially via φ on Y, we can take the fixed points subsheaf

$$f_*^{\ker(\varphi)}\mathcal{F} := (f_*\mathcal{F})^{\ker(\varphi)},$$

and $\lambda_g^{f_*\mathcal{F}}$ restricts, by surjectivity of φ, to a H-equivariant structure on $f_*^{\ker(\varphi)}\mathcal{F}$.

2.2.6. Situation. — From now on we assume that $\mathrm{char}(\Bbbk)$ does not divide the order $\#G$ of G, so that the definitions/equations (2.2.1) and (2.2.2) become meaningful.

2.2.7. **Equivariant derived categories.** — The categories $\mathbf{Coh}(X)^{\mathrm{h}G}$ and $\mathbf{QCoh}(X)^{\mathrm{h}G}$ of (quasi-)coherent equivariant sheaves are abelian categories and the latter has enough injective objects, cf. [Gro57, §5.1], so we can consider their derived categories.

A complex of equivariant sheaves $(\mathcal{F}^\bullet, \lambda^\bullet)$ gives rise to an object $\mathcal{F}^\bullet$ in the derived category $\mathbf{D}^{\mathrm{b}}(X)$ carrying an equivariant object structure induced by $\lambda^\bullet$. So we get a canonical functor

$$\mathbf{D}^{\mathrm{b}}(\mathbf{Coh}(X)^{\mathrm{h}G}) \to \mathbf{D}^{\mathrm{b}}(X)^{\mathrm{h}G}$$

which is an equivalence since $\#G$ is invertible in $\Bbbk$, see [Plo07, §1.1] or more generally [Ela11; Ela14; Che15]. We define the equivariant derived category as

$$\mathbf{D}^{\mathrm{b}}_G(X) \coloneqq \mathbf{D}^{\mathrm{b}}(X)^{\mathrm{h}G}. \tag{2.2.1}$$

2.2.8. — See [LM; OlsASS] as general references for algebraic stacks. The Deligne–Mumford quotient stack $[X/G]$ has an étale atlas $q\colon X \to [X/G]$ given by the canonical quotient map. Denote by $\sigma\colon G \times X \to X$ the action morphism, and by $\mu\colon G \times G \to G$ the multiplication map. Then we have by [OlsASS, Ex. 8.1.12] a pullback square

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\ulcorner\ \ \Swarrow", very near start] & X \arrow[d, "q"] \\ X \arrow[r, "q"] & {[X/G]}, \end{tikzcd}$$

so a descent datum for a sheaf $\mathcal{F}$ on X with respect to q is an isomorphism

$$\lambda\colon \mathrm{pr}_2^*\mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}$$

satisfying the cocycle condition

$$(\mu \times \mathrm{id}_X)^*(\lambda) = \mathrm{pr}_{23}^*(\lambda) \circ (\mathrm{id}_G \times \sigma)^*(\lambda).$$

For the finite group G, such a datum is nothing else than an equivariant structure on $\mathcal{F}$. Following [OlsASS, Ch. 9], the upshot is that, essentially by faithfully flat descent,

$$\mathbf{Coh}(X)^{\mathrm{h}G} \simeq \mathbf{Coh}([X/G]) \quad \text{and} \quad \mathbf{D}^{\mathrm{b}}_G(X) \simeq \mathbf{D}^{\mathrm{b}}([X/G]).$$

From this point of view, the examples and constructions in Example 2.2.4 and ¶2.2.5 arise just as the natural generalizations from the case of schemes to the case of stacks.

2.2.9. — Let $\mathscr{X}$ and $\mathscr{Y}$ be smooth and proper Deligne-Mumford stacks over a field $\Bbbk$ whose characteristic does not divide the orders of the stabilizer groups of $\mathscr{X}$ and $\mathscr{Y}$; the latter property is called *tameness*. Analogous to the situation of varieties discussed in §2.1 we can consider the derived functors associated to push-forward, pull-back, and tensor product of coherent sheaves on stacks, see [LM, Thm. 15.6] and regarding their boundedness in the tame case [OlsASS, Thm. 11.6.5], [Hal22, Thm. 2.1] and [HR15, Thm. C]. For $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(\mathscr{X} \times \mathscr{Y})$ we have the Fourier-Mukai functor

$$\mathrm{FM}_{\mathcal{P}} \coloneqq \mathbf{R}\mathrm{pr}_{\mathscr{Y},*}(\mathbf{L}\mathrm{pr}_{\mathscr{X}}^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{P})\colon \mathbf{D}^{\mathrm{b}}(\mathscr{X}) \to \mathbf{D}^{\mathrm{b}}(\mathscr{Y}).$$

In the case that $\mathscr{X} = [X/G]$ and $\mathscr{Y} = [Y/H]$ are global quotient stacks with, say, X and Y smooth projective and endowed with actions by finite groups G and H, respectively, a concrete explanation of the Fourier-Mukai formalism is given in [BBH, §1.4] and [BL, §8] or, more briefly, in [BKR01, §4] and [Plo05, Ch. 3]. In particular, the Fourier-Mukai functor $\mathrm{FM}_{\mathcal{P}}\colon \mathbf{D}^{\mathrm{b}}_G(X) \to \mathbf{D}^{\mathrm{b}}_H(Y)$ maps $(\mathcal{E}, \lambda)$ to

$$\mathbf{R}\mathrm{pr}^G_{Y,*}(\mathbf{L}\mathrm{pr}^*_X(\mathcal{E}) \overset{\mathbf{L}}{\otimes} \mathcal{P}) \simeq (\mathbf{R}\mathrm{pr}_{Y,*}(\mathbf{L}\mathrm{pr}^*_X(\mathcal{E}) \overset{\mathbf{L}}{\otimes} \mathcal{P}))^G \tag{2.2.2}$$

endowed with a certain equivariant structure. The orbifold version of Orlov's representability theorem (Theorem 2.1.10), see [Kaw04, Thm. 1.1], says that a derived equivalence $\mathbf{D}^{\mathrm{b}}_G(X) \to \mathbf{D}^{\mathrm{b}}_H(Y)$ is (uniquely) represented by a Fourier-Mukai kernel in the category $\mathbf{D}^{\mathrm{b}}_{G\times H}(X \times Y) \simeq \mathbf{D}^{\mathrm{b}}([X/G] \times [Y/H])$.

Let us recall Ploog's method to enhance an invariant derived equivalence to an equivalence of equivariant derived categories. For reference see [Plo07, §§1–2] or in more detail [Plo05, Ch. 3].

2.2.10. Situation. — Let G be a finite group, acting on two smooth projective varieties X and Y. Let us assume that $\mathrm{H}^0(X, \mathcal{O}_X) = \Bbbk$ and $\mathrm{H}^0(Y, \mathcal{O}_Y) = \Bbbk$, e.g. X and Y are geometrically connected, so that Proposition 2.1.13 can be used. We still assume that $\mathrm{char}(\Bbbk)$ does not divide the order $\#G$ of G.

2.2.11. — We consider the following three sets of derived equivalences: First, we have the set of isomorphism classes of derived equivalences between $\mathbf{D}^{\mathrm{b}}(X)$ and $\mathbf{D}^{\mathrm{b}}(Y)$ which commute with the G-action up to isomorphism. In terms of Fourier–Mukai kernels, this is

$$\begin{aligned}&\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))^G \\ &\quad \simeq \{\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(X \times Y) \mid \mathrm{FM}_{\mathcal{P}}\colon \mathbf{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(Y), \text{ and } \forall g \in G\colon (g,g)^*\mathcal{P} \simeq \mathcal{P}\}/{\simeq}.\end{aligned}$$

Second, we have the set of isomorphism classes of derived equivalences between $\mathbf{D}^{\mathrm{b}}_G(X)$ and $\mathbf{D}^{\mathrm{b}}_G(Y)$. These are represented by kernels which are endowed with an equivariant structure for the $(G \times G)$-action on $X \times Y$, cf. ¶2.2.9, so

$$\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}_G(X), \mathbf{D}^{\mathrm{b}}_G(Y)) \simeq \{(\widetilde{\mathcal{P}}, \widetilde{\lambda}) \in \mathbf{D}^{\mathrm{b}}_{G\times G}(X \times Y) \mid \mathrm{FM}_{(\widetilde{\mathcal{P}}, \widetilde{\lambda})}\colon \mathbf{D}^{\mathrm{b}}_G(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_G(Y)\}/{\simeq}.$$

Third, interpolating between the two cases above, we have the set of isomorphism classes of derived equivalences $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(Y)$ which are endowed with an equivariant structure witnessing that Φ "commutes coherently" with the G-action. Again, in terms of kernels, this is

$$\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))^{\mathrm{h}G} := \{(\mathcal{P}, \lambda) \in \mathbf{D}^{\mathrm{b}}_{\Delta G}(X \times Y) \mid \mathrm{FM}_{\mathcal{P}}\colon \mathbf{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(Y)\}/{\simeq},$$

where $\Delta G \subset G \times G$ denotes the diagonal subgroup.

2.2.12. — The sets described in ¶2.2.11 are related to each other by a *forgetful* map and and *inflation* map: The forgetful map

$$\mathrm{for}\colon \mathbf{D}^{\mathrm{b}}_{\Delta G}(X \times Y) \to \mathbf{D}^{\mathrm{b}}(X \times Y)$$

discards equivariant structures, i.e. it maps $(\mathcal{P},\lambda)\mapsto\mathcal{P}$. Note that the kernels in the image of the forgetful map are still G-invariant under the diagonal G-action. The forgetful map can be viewed as an instance of the pullback construction ¶2.2.5.(iii).

For the subgroup $\Delta G\subset G\times G$, or more generally any pair of sub-/supergroup, there is an inflation map

$$\inf_{\Delta G}^{G\times G}\colon\ \mathbf{D}^{\mathrm{b}}_{\Delta G}(X\times Y)\to\mathbf{D}^{\mathrm{b}}_{G\times G}(X\times Y)$$

which maps $(\mathcal{P},\phi)$ to

$$\inf_{\Delta G}^{G\times G}(\mathcal{P},\phi)=\bigoplus_{[g]\in\Delta G\backslash G\times G}g^*\mathcal{P}$$

endowed with a suitable equivariant structure, see [BL, Def. 8.2.1] for details.

We want to apply [Plo07, Thm. 6] not only to autoequivalences but to the sets of equivalences described above, so we spell out the following more general statement of the theorem.

***2.2.13. Theorem* (Ploog).** — *Adopt the setting of Situation 2.2.10, so G is a finite group which acts on two smooth projective varieties X and Y.*

(i) *We have an exact sequence of groups*

$$0\to\mathrm{Hom}(G,\Bbbk^\times)\to\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))^{\mathrm{h}G}\xrightarrow{\text{for}}\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))^{G}\xrightarrow{\delta_X}\mathrm{H}^2(G,\Bbbk^\times)$$

and an exact sequence of pseudo-torsors (i.e. possibly empty torsors)

$$\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X),\mathbf{D}^{\mathrm{b}}(Y))^{\mathrm{h}G}\xrightarrow{\text{for}}\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X),\mathbf{D}^{\mathrm{b}}(Y))^{G}\xrightarrow{\delta_{X,Y}}\mathrm{H}^2(G,\Bbbk^\times).$$

over the respective last three terms of the previous sequence, i.e. the maps are equivariant in the sense of Definition 3.2.10 and $\mathrm{im}(\text{for})=\ker(\delta_{X,Y})$.

(ii) *Assume that G acts faithfully. We have an exact sequence of groups*

$$0\to\mathrm{Z}(G)\to\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))^{\mathrm{h}G}\xrightarrow{\inf_{\Delta G}^{G\times G}}\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}_G(X))$$

and an equivariant map

$$\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X),\mathbf{D}^{\mathrm{b}}(Y))^{\mathrm{h}G}\xrightarrow{\inf_{\Delta G}^{G\times G}}\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}_G(X),\mathbf{D}^{\mathrm{b}}_G(Y)).$$

of pseudo-torsors over the respective last two terms of the previous sequence.

Proof. — The part about groups is exactly [Plo07, Thm. 6]; the part about pseudo-torsors is essentially proven in loc. cit. but not spelled out as such, so we provide a few pointers.

The group structure on $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))^{\mathrm{h}G}$ and its action on $\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X),\mathbf{D}^{\mathrm{b}}(Y))^{\mathrm{h}G}$ are given by convolution of Fourier–Mukai kernels

$$(\mathcal{P},\lambda)\star(\mathcal{P}',\lambda'):=(\mathcal{P}\star\mathcal{P}',\lambda\star\lambda')$$

with $(\lambda\star\lambda')_g:=(\lambda_g\star\lambda'_g)$, which corresponds to composition of associated Fourier–Mukai functors by [Plo07, Lem. 5.(3)], and it is clear that a composition of equivalences is again an equivalence. Now [Plo07, Lem. 5.(5)] provides inverses for kernels of equivalences endowed with an equivariant structure, which lets one deduce that the action is free and transitive.

Similarly the group law on $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))^G$ and its action on $\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X),\mathbf{D}^{\mathrm{b}}(Y))^G$ are given by convolution, and as above one sees that the action is free and transitive. It is clear that $\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}_G(X),\mathbf{D}^{\mathrm{b}}_G(Y))$ is a pseudo-torsor under $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}_G(X))$ since equivalences of categories are invertible.

The description of the actions above settle that the forgetful map is equivariant. The inflation map is equivariant since [Plo07, Lem. 5.(3)] implies that

$$\inf(\mathcal{P},\lambda)\star\inf(\mathcal{P}',\lambda')\simeq\inf((\mathcal{P},\lambda)\star(\mathcal{P}',\lambda')).$$

The map $\delta_{X,Y}$ is defined in [Plo07, Lem. 1], where also the equality $\mathrm{im}(\mathrm{for})=\ker(\delta_{X,Y})$ is proven. The proof in [Plo07, Thm. 6.(2)] that δ_X is a group homomorphism shows more generally that $\delta_{X,Y}$ is equivariant over δ_X. $\square$

2.2.14. — Let us discuss the obstruction maps $\delta_{X,Y}$ from Theorem 2.2.13 in slightly more detail, cf. [Plo07, Lem. 1] or [Plo05, Lem. 3.5]. Let $\mathcal{F}\in\mathbf{D}^{\mathrm{b}}(X)$ be a simple object, i.e. $\mathrm{End}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{F})=\Bbbk$. If $\mathcal{F}$ is G-invariant, there exists some isomorphisms $\lambda_g\colon\mathcal{F}\xrightarrow{\sim}g^*\mathcal{F}$ for $g\in G$, which might not satisfy the cocycle condition required for an equivariant structure. But

$$\delta_{g,h}\coloneqq\lambda_{gh}\circ(h^*\lambda_g\circ\lambda_h)^{-1}\in\Bbbk^\times$$

by simplicity of $\mathcal{F}$, and this provides a 2-cocycle $\delta\colon G\times G\to\Bbbk^\times$, cf. ¶3.1.9, whose class

$$\delta(\mathcal{F})\coloneqq[\delta]\in\mathrm{H}^2(G,\Bbbk^\times)$$

is independent of the choice of the λ_g above. Note that if $[\delta]=0$, then there exists a map $\beta\colon G\to\Bbbk^\times$ certifying that δ is the 2-boundary

$$\delta(g,h)=g.\beta(h)\cdot\beta(g)\cdot\beta(gh)^{-1},$$

and $\lambda'_g\coloneqq\beta(g)\cdot\lambda_g$ will be an equivariant structure.

Later on we will come to the predicament to consider equivariant structures on non-simple sheaves of the form $\mathcal{F}^{\oplus r}$, but where $\mathcal{F}$ is simple. We package the analysis of the obstruction class $\delta(\mathcal{F})$ in such a setting into the following proposition. Afterwards we check that we can use this proposition when G is a symmetric group.

2.2.15. Proposition. — *As before, let G be a finite group, acting on a variety X over $\Bbbk$. Let $\mathcal{F}\in\mathbf{Coh}(X)$ be a simple coherent sheaf on X, i.e. $\mathrm{End}(\mathcal{F})=\Bbbk$, and assume that*

(i) *$\mathcal{F}$ is G-invariant, and*

(ii) *$\mathcal{F}^{\oplus r}$ carries a G-equivariant structure for some integer $r\in\mathbb{N}$.*

If every projective representation of G of dimension r lifts to a linear representation of G,[(2)] *then $\mathcal{F}$ itself admits a G-equivariant structure.*

(2) Here we say that a projective representation $\rho\colon G\to\mathrm{PGL}(V)$ lifts to a linear one if ρ lifts to a homomorphism $\widetilde{\rho}\colon G\to\mathrm{GL}(V)$.

2.2.16. Remark. — Before coming to the proof of Proposition 2.2.15, let us motivate why one would expect the statement, in spite of the brute force nature of its proof below. Consider the short exact sequence

$$1 \to \Bbbk^\times \to \mathrm{GL}(r, \Bbbk) \to \mathrm{PGL}(r, \Bbbk) \to 1$$

and endow its terms with trivial G-actions. Applying non-abelian group cohomology, cf. §3.2, one would like to expect the exact sequence

$$\mathrm{H}^1(G, \mathrm{GL}(r, \Bbbk)) \to \mathrm{H}^1(G, \mathrm{PGL}(r, \Bbbk)) \to \mathrm{H}^2(G, \Bbbk^\times) \xrightarrow{\Delta} \text{“}\,\mathrm{H}^2(G, \mathrm{GL}(r, \Bbbk))\text{”}.$$

The assumption concerning projective representations means that the first map is surjective, so the map Δ is injective. But Δ maps the class which obstructs the existence of an equivariant structure on $\mathcal{E}$ to the obstruction class for $\mathcal{E}^{\oplus r}$, which is trivial by assumption.

The problem is that we need to make sense of the $\mathrm{H}^2(G, \mathrm{GL}(r, \Bbbk))$ term; trying to do so would lead us too far astray into non-abelian group cohomology.

Proof of Proposition 2.2.15. — Since $\mathcal{F}$ is G-invariant, we can pick isomorphisms $\lambda_g \colon \mathcal{F} \xrightarrow{\sim} g^*\mathcal{F}$ for each $g \in G$, and define isomorphisms $\lambda'_g := \lambda_g^{\oplus r} \colon \mathcal{F}^{\oplus r} \xrightarrow{\sim} g^*\mathcal{F}^{\oplus r}$. The obstruction for $(\lambda_g)_g$ to give an equivariant structure is measured by

$$\delta_{g,h} := (h^*\lambda_g \circ \lambda_h)^{-1} \circ \lambda_{gh} \in \Bbbk^\times.$$

Similarly we have the obstruction $\delta'_{g,h} \in \mathrm{GL}(r, \Bbbk)$ for $(\lambda'_g)_g$ to be an equivariant structure; actually we have $\delta'_{g,h} = (\delta_{g,h})^{\oplus r}$. By assumption $\mathcal{F}^{\oplus r}$ admits an equivariant structure $(\lambda''_g)_g$, so we can find elements $\varphi_g \in \mathrm{GL}(r, \Bbbk)$ such that $\lambda''_g = \lambda'_g \circ \varphi_g$.

We claim that the obstruction $\delta''_{g,h}$ of λ'' (which is by assumption just the identity) satisfies

$$\delta'_{g,h} = \varphi_{gh}^{-1} \circ \varphi_g \circ \varphi_h \circ \delta''_{g,h}.$$

Indeed we have by construction

$$\begin{aligned} \lambda''_{gh} &= h^*\lambda''_g \circ \lambda''_h \circ \delta''_{g,h} = h^*(\lambda'_g \circ \varphi_g) \circ \lambda'_h \circ \varphi_h \circ \delta''_{g,h} \\ \lambda''_{gh} &= \lambda'_{gh} \circ \varphi_{gh} = h^*\lambda'_g \circ \lambda'_h \circ \delta'_{g,h} \circ \varphi_{gh}, \end{aligned}$$

so we get $\varphi_g \circ \lambda'_h \circ \varphi_h \circ \delta''_{g,h} = \lambda'_h \circ \delta'_{g,h} \circ \varphi_{gh}$, since the functor h^* is linear. Now the matrices $\varphi_g \in \mathrm{GL}(r, \Bbbk)$ and $\lambda'_h = \mathrm{diag}(\lambda_h, \ldots, \lambda_h)$ commute, so we get

$$\varphi_g \circ \varphi_h \circ \delta''_{g,h} = \delta'_{g,h} \circ \varphi_{gh}.$$

Finally we arrive at the desired equation $\varphi_g \circ \varphi_h \circ \delta''_{g,h} = \varphi_{gh} \circ \delta'_{g,h}$, since $\delta'_{g,h} = \mathrm{diag}(\delta_{g,h}, \ldots, \delta_{g,h})$ commutes with φ_{gh}.

As a consequence, since by assumption $\delta''_{g,h} = \mathrm{id}$ and $\delta'_{g,h} = \mathrm{diag}(\delta_{g,h}, \ldots, \delta_{g,h})$ is diagonal, the map

$$\varphi \colon G \to \mathrm{GL}(r, \Bbbk)$$

is a projective representation of G whose obstruction to being linear is exactly measured by $(\delta_{g,h})_{g,h}$. When the projective representation φ comes from a linear one, the class $[\delta] \in \mathrm{H}^2(G, \Bbbk^\times)$ becomes zero, so $\mathcal{F}$ admits an G-equivariant structure. $\square$

2.2.17. Proposition. — *Assume that $\Bbbk = \overline{\Bbbk}$ is algebraically closed*[(3)]. *Let $G = \mathrm{S}_n$ be a symmetric group and assume $r \in \mathbb{N}$ is odd, then every projective representation of S_n of dimension r is already linear, i.e. the map*

$$\mathrm{Hom}(\mathrm{S}_n, \mathrm{GL}(r, \Bbbk)) \to \mathrm{Hom}(\mathrm{S}_n, \mathrm{PGL}(r, \Bbbk))$$

is surjective.

Proof. — We claim that every non-linear projective representation of S_n is even dimensional. So any representation $\rho\colon \mathrm{S}_n \to \mathrm{PGL}(r, \Bbbk)$, where r is odd by assumption, must be linear.

Schur [Sch11] shows that for $n \geq 4$ the irreducible[(4)] non-linear projective representations of S_n are indexed by strict partitions $(\lambda_i)_i$ of n, that is $\lambda_1 + \cdots + \lambda_\ell = n$ and $\lambda_1 > \cdots > \lambda_\ell$. For $n \leq 3$ there are no irreducible non-linear projective representations of S_n. Their dimension is given by the formula

$$f_\lambda = 2^{\lfloor \frac{n-\ell}{2} \rfloor} g_\lambda \quad \text{with} \quad g_\lambda = \frac{n!}{\lambda_1! \cdots \lambda_\ell!} \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

The number g_λ is in fact an integer since it counts certain “shifted standard tableaux of shape λ”, cf. [MacSF, III.8 Ex. 12]. By the strictness of the partitions, we have $n \geq \ell + 2$, so f_λ is an even integer.

In order to conclude, we use general facts about the projective representations of a finite group G to reduce to the irreducible case, see [KarGR, Ch. 2–3] for details and explanations. Every projective representation $\rho\colon G \to \mathrm{PGL}(r, \Bbbk)$ has an obstruction class $\mathrm{c}(\rho) \in \mathrm{H}^2(G, \Bbbk^\times)$ which measures its failure to be linear. Now any projective representation ρ decomposes into a direct sum of irreducible projective representations ρ_i with the same obstruction class $\mathrm{c}(\rho_i) = \mathrm{c}(\rho)$ as the one of ρ. For the convenience of the reader who is familiar with linear, but not projective, representations of finite groups, we briefly explain the proceeding facts:

For any finite group G, there exists a *Schur cover* $\widetilde{G}$ of G, which is by definition given by an extension

$$0 \to \mathrm{H}_2(G, \mathbb{Z}) \to \widetilde{G} \to G \to 0$$

satisfying $\mathrm{H}_2(G, \mathbb{Z}) \subset \mathrm{Z}(\widetilde{G}) \cap [\widetilde{G}, \widetilde{G}]$. The Schur cover $\widetilde{G}$ has the property that every projective representation ρ of G lifts to a linear representation $\widetilde{\rho}$ of $\widetilde{G}$, i.e. we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}_2(G, \mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{c}(\rho)"] & \widetilde{G} \arrow[r] \arrow[d, "\widetilde{\rho}"] & G \arrow[r] \arrow[d, "\rho"] & 0 \\
1 \arrow[r] & \Bbbk^\times \arrow[r] & \mathrm{GL}(r, \Bbbk) \arrow[r] & \mathrm{PGL}(r, \Bbbk) \arrow[r] & 1.
\end{tikzcd}$$

[(3)] It suffices to assume that $\Bbbk^\times = \Bbbk^{\times 2}$. Then $\mathrm{H}_1(\mathrm{S}_n, \mathbb{Z}) \simeq \mathrm{S}_n^{\mathrm{ab}} \simeq \mathbb{Z}/2\mathbb{Z}$ together with the universal coefficient theorem imply that $\mathrm{H}^2(\mathrm{S}_n, \Bbbk^\times) \to \mathrm{H}^2(\mathrm{S}_n, \overline{\Bbbk}^\times)$ is injective, cf. Proposition 3.1.29.(ii). So a non-linear projective representation of S_n stays non-linear when base-changing from $\Bbbk$ to $\overline{\Bbbk}$.

[(4)] A projective representation $\rho\colon \mathrm{S}_n \to \mathrm{PGL}(r', \Bbbk)$ is *irreducible* if no proper subspace $0 \neq V \subsetneq \Bbbk^{\oplus r'}$ exists which is fixed by every element in $\mathrm{im}(\rho)$.

We have $\mathrm{H}^2(G,\mathbb{Z}) \simeq \mathrm{Hom}(\mathrm{H}_2(G,\mathbb{Z}),\Bbbk^\times)$ by the universal coefficient theorem, cf. Proposition 3.1.29, and the character $\mathrm{c}(\rho)$ in the diagram above becomes the obstruction class mentioned before. After a change of basis of $\Bbbk^{\oplus r}$, which does not affect $\mathrm{c}(\rho)$, one can write $\widetilde{\rho}$ as a direct sum of irreducible representations $\widetilde{\rho}_i\colon \widetilde{G} \to \mathrm{GL}(r_i,\Bbbk)$. Then one sees that each $\widetilde{\rho}_i$ descends to a projective representation $\rho_i\colon G \to \mathrm{PGL}(r_i,\Bbbk)$ with $\mathrm{c}(\rho_i) = \mathrm{c}(\rho)$. □

We now discuss Bridgeland–King–Reid's derived McKay correspondence [BKR01], which allows one to view the derived categories of some equivariant Hilbert schemes, cf. ¶1.1.12, as equivariant derived categories. See also [BBH, §7.6] for a terse account.

2.2.18. Situation. — Let G be a finite group, acting faithfully on a connected, smooth, projective variety X over an algebraically closed field $\Bbbk$ of characteristic 0.

2.2.19. — Recall the equivariant Hilbert scheme $\mathrm{Hilb}^G(X)$ from ¶1.1.12, which is a (irreducible component of the) fine moduli space of G-clusters. Thus there is a universal G-cluster

$$\mathcal{Z} \hookrightarrow \mathrm{Hilb}^G(X) \times X,$$

which is in particular a closed subscheme and it is finite and flat over $\mathrm{Hilb}^G(X)$. Since $\mathcal{Z}$ is a $\{\mathrm{id}\} \times G$-invariant subscheme, its structure sheaf $\mathcal{O}_{\mathcal{Z}}$ is equivariant by Example 2.2.4.(i) and ¶2.2.5.(iv), so

$$\mathcal{O}_{\mathcal{Z}} \in \mathbf{D}^{\mathrm{b}}_{\{\mathrm{id}\}\times G}(\mathrm{Hilb}^G(X) \times X)$$

and we have a Fourier-Mukai functor

$$\mathrm{FM}_{\mathcal{O}_{\mathcal{Z}}}\colon\ \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^G(X)) \to \mathbf{D}^{\mathrm{b}}_G(X)$$

as described in ¶2.2.9. Recall also the Hilbert-Chow morphism $\mathrm{HC}\colon\ \mathrm{Hilb}^G(X) \to X/G$ from ¶1.1.12.

***2.2.20. Theorem* (Derived McKay correspondence).** — *Assume that the canonical sheaf ω_X is locally trivial as a G-equivariant sheaf,*[(5)] *and the dimension condition*

$$\dim(\mathrm{Hilb}^G(X) \times_{X/G} \mathrm{Hilb}^G(X)) \leq \dim(X) + 1 \tag{2.2.3}$$

is satisfied. Then $\mathrm{HC}\colon\ \mathrm{Hilb}^G(X) \to X/G$ is a crepant resolution of singularities and

$$\mathrm{FM}_{\mathcal{O}_{\mathcal{Z}}}\colon\ \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^G(X)) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_G(X)$$

is a derived equivalence. Moreover, whenever X is a symplectic variety, G acts via symplectic automorphisms, and $\mathrm{Hilb}^G(X) \to X/G$ is a priori a crepant resolution, then condition (2.2.3) is satisfied.

Proof. — See [BKR01, Thm. 1.1, Cor. 1.3]. □

(5)The canonical sheaf ω_X is *locally trivial as a G-equivariant sheaf* if every point on X has a G-invariant open neighborhood U on which $(\omega_X, \lambda^{\omega_X})$ is isomorphic to the trivial line bundle $\mathcal{O}_X$ equipped with its canonical G-equivariant structure, cf. Example 2.2.4. In practice this amounts to asking that on U there exists a nowhere vanishing G-invariant n-form, where $n = \dim(X)$.

2.2.21. Proposition. — *Let S be a smooth, projective surface, A an abelian surface, and $n \in \mathbb{N}$. Then we have*

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n(S)) \simeq \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^{\mathrm{S}_n}(S^{\times n})) \simeq \mathbf{D}^{\mathrm{b}}_{\mathrm{S}_n}(S^{\times n})$$

and

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^{n-1}(A)) \simeq \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^{\mathrm{S}_n}(A \otimes \Gamma_n)) \simeq \mathbf{D}^{\mathrm{b}}_{\mathrm{S}_n}(A \otimes \Gamma_n).$$

Proof. — The derived equivalences on the left hand sides are induced from isomorphisms of varieties and were discussed in ¶1.1.12 and Proposition 1.1.14, respectively. The equivalences on the right hand sides come from the derived McKay correspondence (Theorem 2.2.20); we check its assumptions first for the case of Hilbert schemes of points and afterwards for the case of generalized Kummer varieties.

Assume that S has a trivial canonical sheaf $\omega_S \simeq \mathcal{O}_S$, so S is a symplectic variety, say with symplectic form $\sigma \in \mathrm{H}^0(S, \Omega_S^2)$. Then also $S^{\times n}$ is a symplectic variety, with symplectic form

$$\sigma' := \mathrm{pr}_1^*\sigma + \cdots + \mathrm{pr}_n^*\sigma,$$

and it becomes clear that S_n acts via symplectic automorphisms. The Pfaffian

$$\mathrm{pf}(\sigma') := \frac{1}{n!}\sigma'^{\wedge n} = \mathrm{pr}_1^*\sigma \wedge \cdots \wedge \mathrm{pr}_n^*\sigma \in \mathrm{H}^0(S^{\times n}, \omega_{S^{\times n}})$$

is then also S_n-invariant and nowhere vanishing, since the square of the Pfaffian is the determinant of the symplectic form σ'. So $\omega_{S^{\times n}}$ is trivial as an equivariant sheaf. We already know that the Hilbert-Chow morphism $\mathrm{Hilb}^{\mathrm{S}_n}(S^{\times n}) \to \mathrm{Sym}^n(S)$ is a crepant resolution, cf. Example 1.1.9 and ¶1.1.12, so all conditions of Theorem 2.2.20 are satisfied.

As explained in [Plo07, §3] one deduces the result for smooth projective surfaces by checking condition (2.2.3) locally on S, so that one can assume without loss of generality $\omega_S \simeq \mathcal{O}_S$.

Regarding generalized Kummer varieties, we have seen in Proposition 1.1.15 that

$$\mathrm{Hilb}^{\mathrm{S}_n}(A \otimes \Gamma_n) \to (A \otimes \Gamma_n)/\mathrm{S}_n$$

is a crepant resolution. In the proof of this fact we have exhibited a S_n-invariant volume form $\omega \in \mathrm{H}^0(A \otimes \Gamma_n, \omega_{A\otimes\Gamma_n})$, cf. (1.1.2), so $\omega_{A\otimes\Gamma_n}$ is trivial as an equivariant sheaf. The form ω is the Pfaffian of the S_n-invariant 2-form

$$\mathrm{d}z_1 \wedge \mathrm{d}z_1' + \cdots + \mathrm{d}z_{n-1} \wedge \mathrm{d}z_{n-1}',$$

with notation as in Proposition 1.1.15, so the latter is non-degenerate, as desired. □

2.3. Derived equivalences of abelian varieties and Kummer surfaces

The study of derived equivalences of abelian varieties was established by Mukai [Muk81; Muk98], Polishchuk [Pol96], and Orlov [Orl02]. For an exposition of the theory see [HuyFM, Ch. 9]. See [PolAV, Ch. 11, Ch. 15] for Polishchuk's viewpoint. In this section we mainly focus on Orlov's construction in the setting of abelian varieties, and afterwards we briefly discuss derived equivalences of Kummer surfaces, following [Plo07] and [HLOY03]. In addition to this summary and survey, we spell out some computations about groups of symplectic isomorphisms of an abelian variety, which were certainly known to Mukai.

***2.3.1. Theorem* (Mukai).** — *Let A be an abelian variety over an algebraically closed field $\Bbbk$, then A and its dual abelian variety $A^\vee$ are derived equivalent,*

$$\mathbf{D}^{\mathrm{b}}(A) \simeq \mathbf{D}^{\mathrm{b}}(A^\vee).$$

Proof. — See [Muk81, Thm. 2.2]. The derived equivalence constructed by Mukai is given by the Fourier–Mukai functor

$$\mathrm{FM}_{\mathcal{P}} \colon \mathbf{D}^{\mathrm{b}}(A) \to \mathbf{D}^{\mathrm{b}}(A^\vee),$$

where $\mathcal{P} \in \mathrm{Pic}(A \times A^\vee)$ is the Poincaré bundle, cf. ¶1.2.8. It is shown that

$$\mathrm{FM}_{\mathcal{P}} \circ \mathrm{FM}_{\mathcal{P}^{\mathrm{t}}} = [-1]_{A^\vee,*} \circ [-g] \quad \text{and} \quad \mathrm{FM}_{\mathcal{P}^{\mathrm{t}}} \circ \mathrm{FM}_{\mathcal{P}} = [-1]_{A,*} \circ [-g],$$

where $g \coloneqq \dim(A)$, and $\mathcal{P}^{\mathrm{t}}$ is the Poincaré bundle on $A^\vee \times A$, and $[-1]_A \colon A \to A$ denotes the negation involution of A. $\square$

Mukai, Polishchuk, and later Orlov, introduced and studied the following group of "symplectic" isomorphisms in order to describe derived equivalences of abelian varieties.

***2.3.2. Definition* (Symplectic isomorphisms).** — Let A and B be abelian varieties. For a homomorphism $f \colon A \times A^\vee \to B \times B^\vee$ of abelian varieties we write

$$f = \begin{pmatrix} f_1 \colon A \to B & f_2 \colon A^\vee \to B \\ f_3 \colon A \to B^\vee & f_4 \colon A^\vee \to B^\vee \end{pmatrix} \quad \text{and} \quad \tilde{f} \coloneqq \begin{pmatrix} f_4^\vee & -f_2^\vee \\ -f_3^\vee & f_1^\vee \end{pmatrix},$$

where we have implicitly identified A with $A^{\vee\vee}$, and B with $B^{\vee\vee}$ via (1.2.1). Denote by

$$\mathrm{Sp}(A,B) \coloneqq \{f \colon A \times A^\vee \xrightarrow{\sim} B \times B^\vee \mid \tilde{f} = f^{-1}\}$$

the set of *symplectic isomorphisms*, and define the *Mukai–Polishchuk group* as $\mathrm{Sp}(A) \coloneqq \mathrm{Sp}(A,A)$. We call a symplectic isomorphism f *admissible* if any f_i is an isogeny.

2.3.3. Remark. — A symplectic isomorphism $f \in \mathrm{Sp}(A,B)$ is admissible if any component f_j is zero, since then some other component f_i has to be an isomorphism.

***2.3.4. Remark* (on viewpoint and notation).** — Mukai and Orlov use the notation $\mathrm{U}(A \times A^\vee)$ instead of $\mathrm{Sp}(A)$ and call elements of it "unitary" and "isometries", respectively. Polishchuk considers the notion of symplectic biextensions and thus suggests to call elements of $\mathrm{Sp}(A)$ symplectic. As discussed in [LT17, §1], over the complex numbers, the group $\mathrm{Sp}(A)$ can be realized as a unitary group as well as a symplectic group, but Polishchuk's symplectic viewpoint works over arbitrary fields.

Over the complex numbers, one can also describe the elements of $\mathrm{Sp}(A)$ as Hodge isometries

$$\mathrm{H}^1(A^{\mathrm{an}}, \mathbb{Z}) \oplus \mathrm{H}^1(A^{\mathrm{an}}, \mathbb{Z})^\vee \to \mathrm{H}^1(A^{\mathrm{an}}, \mathbb{Z}) \oplus \mathrm{H}^1(A^{\mathrm{an}}, \mathbb{Z})^\vee,$$

where both sides are endowed with the pairing $q(a, \alpha) := 2\alpha(a)$, cf. [HuyFM, Cor. 9.50].

We write $\mathrm{Sp}(A)$ instead of Polishchuk's $\mathrm{Sp}(A \times A^\vee)$, since the latter would be too heavy notation for us later on, so we prefer the shorter form. The following propositions, which we will synthesize in Proposition 5.1.9, affirm the terminology "symplectic".

2.3.5. Proposition. — *Let A be a principally polarizable abelian variety which satisfies $\mathrm{End}(A) = \mathbb{Z}$. Then, for $n \in \mathbb{N}$,*

$$\mathrm{Sp}(A^{\times n}) \simeq \mathrm{Sp}(2n, \mathbb{Z})$$

is a classical symplectic group.

Proof. — Denote by $\lambda_0 \colon A \xrightarrow{\sim} A^\vee$ the (unique) principal polarization of A, and let $f \in \mathrm{Sp}(A^{\times n}) \simeq \mathrm{Sp}(A \otimes \mathbb{Z}^n)$. Then, by the assumption $\mathrm{End}(A) = \mathbb{Z}$, we can write

$$\begin{aligned}
f_1 &= \mathrm{id} \otimes g_1 \in \mathrm{Hom}(A, A) \otimes \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}^n)\\
f_2 &= \lambda_0^{-1} \otimes g_2 \in \mathrm{Hom}(A^\vee, A) \otimes \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{n\vee}, \mathbb{Z}^n)\\
f_3 &= \lambda_0 \otimes g_3 \in \mathrm{Hom}(A, A^\vee) \otimes \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}^{n\vee})\\
f_4 &= \mathrm{id} \otimes g_4 \in \mathrm{Hom}(A^\vee, A^\vee) \otimes \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{n\vee}, \mathbb{Z}^{n\vee}).
\end{aligned}$$

Taking the standard basis of $\mathbb{Z}^n$ and its dual basis of $\mathbb{Z}^{n\vee}$, we can view each g_i as a matrix $M_i \in \mathrm{Mat}(n \times n, \mathbb{Z})$. Since $\lambda_0^{-1} \circ \lambda_0 = \mathrm{id}$ and $\lambda_0 \circ \lambda_0^{-1} = \mathrm{id}$, multiplication in the group $\mathrm{Sp}(A^{\times n})$ corresponds to matrix multiplication, so we get a group homomorphism

$$\begin{aligned}
\mathrm{Sp}(A^{\times n}) &\to \mathrm{Mat}(2n \times 2n, \mathbb{Z}) \qquad (2.3.1)\\
f &\mapsto \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}.
\end{aligned}$$

Let us compute the matrix corresponding to $\widetilde{f}$. Recall that $g_i^\vee$ corresponds to the transposed matrix M_i^{t}. Now, identifying A with $A^{\vee\vee}$ implicitly, we get by definition, substitution, and since polarizations are symmetric, cf. Definition 1.2.17, that

$$\tilde{f} = \begin{pmatrix} f_4^\vee & -f_2^\vee \\ -f_3^\vee & f_1^\vee \end{pmatrix} = \begin{pmatrix} \mathrm{id}^\vee \otimes g_4^\vee & -(\lambda_0^{-1})^\vee \otimes g_2^\vee \\ -\lambda_0^\vee \otimes g_3^\vee & \mathrm{id}^\vee \otimes g_1^\vee \end{pmatrix} = \begin{pmatrix} \mathrm{id} \otimes g_4^\vee & -(\lambda_0^{-1}) \otimes g_2^\vee \\ -\lambda_0 \otimes g_3^\vee & \mathrm{id} \otimes g_1^\vee \end{pmatrix},$$

which corresponds to the matrix

$$\begin{pmatrix} M_4^{\mathrm{t}} & -M_2^{\mathrm{t}} \\ -M_3^{\mathrm{t}} & M_1^{\mathrm{t}} \end{pmatrix} = J^{\mathrm{t}} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}^{\mathrm{t}} J \qquad \text{where} \quad J = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}.$$

This means that the condition $\widetilde{f} = f^{-1}$ singles out symplectic matrices. In conclusion, the map in (2.3.1) provides the desired isomorphism. ☐

2.3.6. Definition. — The *Hecke congruence subgroup of level* $l \in \mathbb{N} \setminus \{0\}$ is defined as

$$\Gamma_0(l) := \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; a_3 \equiv 0 \mod l \right\}.$$

2.3.7. Proposition. — *Let A be an abelian variety satisfying* $\mathrm{End}(A) = \mathbb{Z}$. *Write* $\mathrm{Hom}(A, A^\vee) = \mathbb{Z} \cdot \lambda_0$ *and* $\mathrm{Hom}(A^\vee, A) = \mathbb{Z} \cdot \lambda_0'$, *where λ_0 and λ_0' are polarizations, and define $l \in \mathbb{N}$ by $\lambda_0 \circ \lambda_0' = [l]$. Then we have an isomorphism*

$$\mathrm{Sp}(A) \simeq \Gamma_0(l) \subset \mathrm{SL}(2,\mathbb{Z})$$

with the Hecke congruence subgroup of level l.

Proof. — Let $f \in \mathrm{Sp}(A)$, then we can write

$$f = \begin{pmatrix} a_1 \cdot \mathrm{id} & a_2 \cdot \lambda_0' \\ a_3 \cdot \lambda_0 & a_4 \cdot \mathrm{id} \end{pmatrix}$$

for some $a_i \in \mathbb{Z}$. Since $\lambda_0' \circ \lambda_0 = [l]$ and $\lambda_0 \circ \lambda_0' = [l]$, multiplication in $\mathrm{Sp}(A)$ is not quite matrix multiplication. Instead we get a group homomorphism

$$\begin{aligned} \mathrm{Sp}(A) &\to \mathrm{Mat}(2 \times 2, \mathbb{Z}) \\ f &\mapsto \begin{pmatrix} a_1 & a_2 \\ l \cdot a_3 & a_4 \end{pmatrix}. \end{aligned} \tag{2.3.2}$$

The same calculation of $\widetilde{f}$ as in the proof of Proposition 2.3.5 explains that the condition $\widetilde{f} = f^{-1}$ singles out symplectic matrices. But $\mathrm{Sp}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$, so the map (2.3.2) provides the desired isomorphism. □

Next we discuss Orlov's fundamental short exact sequence which describes derived equivalences of abelian varieties in terms of symplectic isomorphisms.

***2.3.8. Theorem* (Orlov).** — *Let A and B be two abelian varieties over an algebraically closed field $\Bbbk$ of characteristic* 0, *then we have a short exact sequence of groups*

$$0 \to \mathbb{Z} \times A(\Bbbk) \times A^\vee(\Bbbk) \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A)) \xrightarrow{\gamma_A} \mathrm{Sp}'(A) \to 0, \tag{2.3.3}$$

and a surjective map

$$\gamma_{A,B} \colon \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A), \mathbf{D}^{\mathrm{b}}(B)) \twoheadrightarrow \mathrm{Sp}'(A,B),$$

where $\mathrm{Sp}'(A) \subset \mathrm{Sp}(A)$, *respectively* $\mathrm{Sp}'(A,B) \subset \mathrm{Sp}(A,B)$, *is a subgroup/subset which contains all admissible symplectic isomorphisms.*[6]

Moreover, the maps γ_A and $\gamma_{A,B}$ are compatible in the sense that

$$\gamma_{A,B}(\Phi' \circ \Phi) = \gamma_{A,B}(\Phi')\gamma_A(\Phi) \tag{2.3.4}$$

for every $\Phi \in \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A))$ *and* $\Phi' \in \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A), \mathbf{D}^{\mathrm{b}}(B))$.

---

(6) According to Orlov we should have $\mathrm{Sp}'(A,B) = \mathrm{Sp}(A,B)$, but see Remark 2.3.14.

2.3.9. Notation. — We will use the notation

$$\mathrm{Sp}'(A,B) := \mathrm{im}(\gamma_{A,B}\colon \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A),\mathbf{D}^{\mathrm{b}}(B)) \to \mathrm{Sp}(A,B)).$$

Proof of Theorem 2.3.8. — See [Orl02, Thm. 4.14, Prop. 4.11, Prop. 4.12]. Let us also mention Polishchuk's work [Pol96] and the exposition in [HuyFM, §§9.4–9.5].

The left hand side inclusion in (2.3.3) arises as the composition of the standard autoequivalence inclusion (2.1.4) with the homomorphism

$$A(\Bbbk) \times A^\vee(\Bbbk) \hookrightarrow \mathrm{Aut}(A) \ltimes \mathrm{Pic}(A),$$
$$(a,\alpha) \mapsto (\mathrm{t}_a, \mathcal{P}_\alpha)$$

where the product on the left hand side is indeed a direct product, since line bundles in $A^\vee(\Bbbk)$ are by definition translation invariant. So $n \in \mathbb{Z}$ is mapped to the shift functor $[n]$, and $a \in A$ is mapped to the push-forward $(\mathrm{t}_a)_*$ along the translation morphism t_a, and $\alpha \in A^\vee$ is mapped to the twist functor $\mathcal{P}_\alpha \otimes -$.[(7)]

We reproduce some details below in ¶2.3.10 about the definition of the maps γ_A and $\gamma_{A,B}$ and discuss the construction witnessing their surjectivity in Construction 2.3.12. □

2.3.10. — Let A and B be abelian varieties over any field $\Bbbk$. The definition of the map $\gamma_{A,B}$ involves replacing an equivalence $\Phi\colon \mathbf{D}^{\mathrm{b}}(A) \to \mathbf{D}^{\mathrm{b}}(B)$ by an equivalence $F_\Phi\colon \mathbf{D}^{\mathrm{b}}(A\times A^\vee) \to \mathbf{D}^{\mathrm{b}}(B\times B^\vee)$ which is closer to geometry in the sense that it can be realized as push-forward along an isomorphism $\gamma_{A,B}(\Phi)$ followed by a line bundle twist. Let $\mathcal{E} \in \mathbf{D}^{\mathrm{b}}(A\times B)$ be the Fourier–Mukai kernel of the equivalence Φ, and set $\mathcal{E}^{\mathrm{R}} := \mathcal{E}^\vee[\dim(A)] := \mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{O}_{A\times B})[\dim(A)]$, which is again a kernel of an equivalence, since it is the transpose[(8)] of the inverse of Φ. Now F_Φ is defined as the composition of equivalences

$$\begin{tikzcd}
\mathbf{D}^{\mathrm{b}}(A\times A^\vee) \arrow[rr, dashed, "F_\Phi"] \arrow[d, "(\mathrm{id},\mathrm{FM}_{\mathcal{P}_A})"'] && \mathbf{D}^{\mathrm{b}}(B\times B^\vee) \\
\mathbf{D}^{\mathrm{b}}(A\times A) \arrow[d, "(+_A,\mathrm{id})_*"'] && \mathbf{D}^{\mathrm{b}}(B\times B) \arrow[u, "(\mathrm{id},\mathrm{FM}_{\mathcal{P}_B})^{-1}"'] \\
\mathbf{D}^{\mathrm{b}}(A\times A) \arrow[rr, "(\mathrm{FM}_{\mathcal{E}},\mathrm{FM}_{\mathcal{E}^{\mathrm{R}}})"] && \mathbf{D}^{\mathrm{b}}(B\times B), \arrow[u, "(+_B,\mathrm{id})_*^{-1}"']
\end{tikzcd} \tag{2.3.5}$$

cf. [Orl02, Def. 2.9] and [HuyFM, Def. 9.34].

Consider the autoequivalence $\Phi_{(a,\alpha)} := \mathrm{t}_{a,*} \circ (\mathcal{P}_\alpha \otimes -) \simeq (\mathcal{P}_\alpha \otimes -) \circ \mathrm{t}_{a,*}$ for a rational point $(a,\alpha) \in A \times A^\vee$. The vertical map on the left side of (2.3.5) assembles all these autoequivalences together by mapping the skyscraper sheaf $\mathrm{k}(-a,\alpha)$ to the kernel $\mathcal{O}_{\Gamma_{\mathrm{t}_a}} \otimes \mathrm{pr}_1^*\mathcal{P}_\alpha$ of the autoequivalence $\Phi_{(a,\alpha)}$, cf. [Orl02, Ass. 2.8]. The horizontal

(7) This differs from Ploog's convention in [Plo05] by a sign, where $a \in A$ is mapped instead to the pullback functor $(\mathrm{t}_a)^* = (\mathrm{t}_{-a})_*$.

(8) Given a kernel $\mathcal{E} \in \mathbf{D}^{\mathrm{b}}(A\times B)$, its *transpose* $\mathcal{E}^{\mathrm{t}} \in \mathbf{D}^{\mathrm{b}}(B\times A)$ is the object corresponding to $\mathcal{E}$ under the equivalence $\mathbf{D}^{\mathrm{b}}(A\times B) \simeq \mathbf{D}^{\mathrm{b}}(B\times A)$ induced by the canonical identification $A\times B \xrightarrow{\sim} B\times A$.

map at the bottom of (2.3.5) maps a Fourier–Mukai kernel $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(A \times A)$ to the kernel of the conjugated functor

$$\mathrm{FM}_{\mathcal{E}^{\mathrm{t}}}^{-1} \circ \mathrm{FM}_{\mathcal{F}} \circ \mathrm{FM}_{\mathcal{E}^{\mathrm{t}}} \colon \mathbf{D}^{\mathrm{b}}(B) \to \mathbf{D}^{\mathrm{b}}(B),$$

where $\mathcal{E}^{\mathrm{t}}$ is the transpose of $\mathcal{E}$, cf. [Orl02, Lem. 1.6].

As promised, F_{Φ} maps skyscraper sheaves to skyscraper sheaves and hence is of the form

$$F_{\Phi} \simeq (\mathcal{L}_{\Phi} \otimes -) \circ \gamma_{A,B}(\Phi)_*$$

for some line bundle $\mathcal{L}_{\Phi} \in \mathrm{Pic}(B \times B^{\vee})$ and isomorphism

$$\gamma_{A,B}(\Phi) \colon A \times A^{\vee} \to B \times B^{\vee}$$

of abelian varieties, cf. [Orl02, Thm. 2.10]. The map $\gamma_{A,B}$ factors over $\mathrm{Sp}(A, B) \subset \mathrm{Isom}(A \times A^{\vee}, B \times B^{\vee})$ by [Orl02, Prop. 2.18], and the compatibility (2.3.4) with composition holds by [Orl02, Prop. 2.15].

The last property we want to mention is that we have $\gamma_{A,B}(\Phi)(a, \alpha) = (b, \beta)$ if and only if

$$\Phi_{(b,\beta)} \circ \Phi \simeq \Phi \circ \Phi_{(a,\alpha)}, \tag{2.3.6}$$

cf. [Orl02, Cor. 2.13] or [HuyFM, Cor. 9.44].

2.3.11. Example. — We summarize [HuyFM, Ex. 9.38] and [Plo05, Ex. 4.5][9].

(i) For $\mathcal{L} \in \mathrm{Pic}(A)$ we have

$$\gamma_A(\mathcal{L} \otimes -) = \begin{pmatrix} \mathrm{id} & 0 \\ \varphi_{\mathcal{L}} & \mathrm{id} \end{pmatrix}.$$

So when $\mathcal{L} \in \mathrm{Pic}^0(A)$, we get $\gamma_A(\mathcal{L} \otimes -) = \mathrm{id}$, as Orlov's theorem claims.

(ii) Let $a \in A$ and $f \in \mathrm{Isom}_{\mathrm{AV}}(A, B)$ an isomorphism, then $\gamma_A((\mathrm{t}_a)_*) = \mathrm{id}$ and

$$\gamma_{B,A}(f^*) = \begin{pmatrix} f^{-1} & 0 \\ 0 & f^{\vee} \end{pmatrix}.$$

(iii) For the Poincaré bundle $\mathcal{P} \in \mathrm{Pic}(A \times A^{\vee})$ we have

$$\gamma_{A,A^{\vee}}(\mathrm{FM}_{\mathcal{P}}) = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}.$$

2.3.12. Construction. — We summarize the steps of Orlov's construction [Orl02, Constr. 4.10] concerning the image of the map $\gamma_{A,B} \colon \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A), \mathbf{D}^{\mathrm{b}}(B)) \to \mathrm{Sp}(A, B)$ in Theorem 2.3.8. The construction makes use of Mukai's theory of semi-homogeneous vector bundles, which we recalled in §1.3, and consists of the following steps:

(1) Consider a symplectic isomorphism

$$f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in \mathrm{Sp}(A, B)$$

and assume that the map $f_2 \colon A^{\vee} \to B$ is an isogeny.

[9] Beware of a misprint in [Plo05, Ex. 4.5.(3)] where f^* was mistyped as f_*.

(2) Denote by f_2^{-1} the inverse isogeny of f_2 with rational coefficients, and subsequently define the map $g \in \operatorname{Hom}(A \times B, A^\vee \times B^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$g := \begin{pmatrix} f_2^{-1} \circ f_1 & -f_2^{-1} \\ -(f_2^{-1})^\vee & f_4 \circ f_2^{-1} \end{pmatrix}.$$

Its dual morphism is

$$g^\vee := \begin{pmatrix} f_1^\vee \circ (f_2^{-1})^\vee & (-f_2^{-1})^{\vee\vee} \\ -(f_2^{-1})^\vee & (f_2^{-1})^\vee \circ f_4^\vee \end{pmatrix}.$$

(3) Since f is symplectic, we have $-f_1 \circ f_2^\vee + f_2 \circ f_1^\vee = 0$ and $f_4^\vee \circ f_2 - f_2^\vee \circ f_4 = 0$ by multiplying out matrices while suppressing evaluation isomorphisms (1.2.1). So we see that g is symmetric, i.e. $g = g^\vee \circ \mathrm{ev}$. Hence g is contained in the image of the injection

$$\mathrm{NS}(A \times B) \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow \operatorname{Hom}(A \times B, A^\vee \times B^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$$

which associates to a line bundle $\mathcal{L}$ the map $\varphi_{\mathcal{L}}(x) = \mathrm{t}_x^*\mathcal{L} \otimes \mathcal{L}^\vee$, cf. ¶1.2.11. So the map g corresponds to an element

$$\mu := [\mathcal{L}] \otimes \frac{1}{\ell} \in \mathrm{NS}(A \times B) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Then Orlov takes a simple semi-homogeneous vector bundle $\mathcal{E}$ on $A \times B$ of slope μ, and considers it as a Fourier–Mukai kernel. Following Mukai, a construction of $\mathcal{E}$ is given by the following steps:

(4) By Proposition 1.3.7.(i), the sheaf

$$\mathcal{F} := [\ell]_*(\mathcal{L}^{\otimes \ell})$$

is a semi-homogeneous vector bundle on $A \times B$ of slope $\mu(\mathcal{F}) = \mu$.

(5) Consider a Jordan–Hölder filtration $0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}$, where each

$$\mathcal{E}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$$

is a *simple* semi-homogeneous vector bundle of slope $\mu(\mathcal{E}_i) = \mu$, cf. Proposition 1.3.7.(ii). Finally, take any of the vector bundles $\mathcal{E}_i$ as the kernel of a Fourier–Mukai functor

$$\mathrm{FM}_{\mathcal{E}_i} \colon \mathbf{D}^{\mathrm{b}}(A) \to \mathbf{D}^{\mathrm{b}}(B).$$

Then Orlov shows in [Orl02, Prop. 4.11, Prop. 4.12] using the theory of semi-homogeneous vector bundles that the Fourier–Mukai functor $\mathrm{FM}_{\mathcal{E}_i}$ is a desired derived equivalence satisfying

$$\gamma_{A,B}(\mathrm{FM}_{\mathcal{E}_i}) = f.$$

2.3.13. Remark. — Forgetting about f_3 in step (2) of Construction 2.3.12 is not an issue. Indeed, since f is a symplectic isomorphism we have $-f_3 \circ f_2^\vee + f_4 \circ f_1^\vee = \mathrm{id}$, and since f_2 is assumed to be an isogeny, this determines f_3 uniquely.

The arbitrary choice of one of the graded pieces $\mathcal{E}_i$ in step (5) of Construction 2.3.12 is also no reason for concern, since the kernel in Orlov's sequence (2.3.3) allows for some flexibility when constructing a preimage of some symplectic isomorphism.

2.3.14. Remark. — In Construction 2.3.12 one assumes that f_2 is an isogeny, and Orlov claims without proof that a symplectic isomorphism can always be factored into the composition of two symplectic isomorphisms which satisfy this extra condition, cf. [Orl02, p. 591]. But it is not clear how to perform such a factorization in general. The article [LT17] focuses on this gap in Orlov's proof and addresses the issue for simple abelian varieties. Let us explain that every admissible symplectic isomorphism lies in the image.

2.3.15. Proposition. — *The image* $\mathrm{Sp}'(A,B) \coloneqq \mathrm{im}(\gamma_{A,B}) \subset \mathrm{Sp}(A,B)$ *contains all admissible symplectic isomorphism in the sense of Definition 2.3.2. In particular, if A is a simple abelian variety, then we have an equality* $\mathrm{Sp}'(A,B) = \mathrm{Sp}(A,B)$.

Proof. — Let $f \in \mathrm{Sp}(A,B)$ be a symplectic isomorphism and write

$$f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}.$$

(a) By Construction 2.3.12 we know that $f \in \mathrm{im}(\gamma_{A,B})$ if $f_2 \colon A^\vee \to B$ is an isogeny.

(b) Just as an exercise with the constructions encountered so far, let us spell out the case $f_2 = 0$ without appealing to case (c) below. If $f_2 = 0$, we know that $f_1 \colon A \to B$ is an isomorphism with inverse $f_4^\vee$. Using Example 2.3.11, we see that

$$\gamma_{B,A}(f_1^*)f = \begin{pmatrix} f_1^{-1} & 0 \\ 0 & f_1^\vee \end{pmatrix} \begin{pmatrix} f_1 & 0 \\ f_3 & f_1^{-1,\vee} \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ f_1^\vee f_3 & \mathrm{id} \end{pmatrix}. \tag{2.3.7}$$

We know that $f_1^\vee f_3 \colon A \to A^\vee$ is a symmetric homomorphism, since f is symplectic. So there exists a line bundle $\mathcal{L} \in \mathrm{Pic}(A)$ such that $f_1^\vee f_3 = \varphi_{\mathcal{L}}$, cf. ¶1.2.11. Then the right hand side of (2.3.7) is $\gamma_A(\mathcal{L} \otimes -)$ by Example 2.3.11, and we get

$$f = \gamma_{B,A}(f_1^*)^{-1}\gamma_A(\mathcal{L} \otimes -) \in \mathrm{im}(\gamma_{A,B}).$$

(c) Assume that some f_i is an isogeny. Let $\mathcal{P}_A \in \mathrm{Pic}(A \times A^\vee)$ be the Poincaré bundle of A, and let $\mathcal{P}_B$ be the Poincaré bundle of B. We know that

$$\gamma_{A,A^\vee}(\mathrm{FM}_{\mathcal{P}_A}) = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$$

by Example 2.3.11. Then pre- and/or post-composition with $\gamma_{A^\vee,A}(\mathrm{FM}_{\mathcal{P}_A}^{-1})$ and $\gamma_{B,B^\vee}(\mathrm{FM}_{\mathcal{P}_B})$ allows to rearrange the matrix entries of f so that we can reduce to case (a) or (b).

Finally, we consider a simple abelian variety A. Without loss of generality, we can assume that $\mathrm{Sp}(A,B)$ is non-empty; in particular $\dim(A) = \dim(B)$. So a homomorphism $f \colon A \to B$ is an isogeny if and only if $\ker(f)$ is finite. But by simplicity of A, either $\ker(f) = A$ or $\dim(\ker(f)) = 0$, as desired. □

We now come to the discussion of derived equivalences of Kummer K3 surfaces, cf. Example 1.1.6.

***2.3.16. Theorem* (Hosono–Lian–Oguiso–Yau).** — *Let A and B be abelian surfaces over the field $\mathbb{C}$ of complex numbers, then we have*

$$\mathbf{D}^{\mathrm{b}}(A) \simeq \mathbf{D}^{\mathrm{b}}(B) \quad \textit{if and only if} \quad \mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^1(A)) \simeq \mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^1(B)),$$

and, even stronger, this is also equivalent to $\mathrm{Kum}^1(A) \simeq \mathrm{Kum}^1(B)$.

Proof. — See [HLOY03, Thm. 0.1]. The argument uses the derived Torelli theorem for K3 surfaces and for abelian surfaces, cf. [BM01, Thm. 5.1], which classifies derived equivalences in terms of Hodge isometries of transcendental lattices. The stronger statement is due to the fact that Kummer K3 surfaces have no non-trivial Fourier–Mukai partners, as discussed in ¶2.1.14. □

By Remark 1.1.17 a Kummer surface is exactly a generalized Kummer variety $\mathrm{Kum}^1(A)$ of dimension 2. So by Proposition 2.2.21 its derived category can be described as an equivariant derived category,

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^1(A)) \simeq \mathbf{D}^{\mathrm{b}}_{\langle -1\rangle}(A) \simeq \mathbf{D}^{\mathrm{b}}([A/\langle -1\rangle]). \tag{2.3.8}$$

***2.3.17. Theorem* (Stellari).** — *Let A and B be abelian varieties (of any dimension) over an algebraically closed field $\Bbbk$ of characteristic 0. Then*

$$\mathbf{D}^{\mathrm{b}}(A) \simeq \mathbf{D}^{\mathrm{b}}(B) \quad \textit{implies} \quad \mathbf{D}^{\mathrm{b}}_{\langle -1\rangle}(A) \simeq \mathbf{D}^{\mathrm{b}}_{\langle -1\rangle}(B).$$

Proof. — See [Ste07, Thm. 1.1]. Analyzing the map $\gamma_{A,B}$ in Theorem 2.3.8 with respect to the involution action by $[-1]_A \colon A \to A$ yields a surjection

$$\gamma_{A,B} \colon \operatorname{Eq}(\mathbf{D}^{\mathrm{b}}(A), \mathbf{D}^{\mathrm{b}}(B))^{\langle -1\rangle} \twoheadrightarrow \mathrm{Sp}'(A,B),$$

as explained in [Ste07, Prop. 3.1]. By assumption there exists a derived equivalence between A and B, which witnesses that the set of symplectic isomorphisms on the right is non-empty. Since the Schur multiplier $\mathrm{H}^2(\mathbb{Z}/2\mathbb{Z}, \Bbbk^\times) = 0$ is trivial, cf. Example 3.1.30, one can then employ Ploog's method (Theorem 2.2.13) to enhance an invariant derived equivalence to a derived equivalence of equivariant categories. □

Ploog himself focuses on the case of derived autoequivalences in [Plo07] and obtains the following theorem about Kummer surfaces.

***2.3.18. Theorem* (Ploog).** — *Let A be an abelian surface over an algebraically closed field $\Bbbk$ of characteristic 0. Then we have a short exact sequence*

$$0 \to \mathbb{Z} \times A[2] \times A^\vee[2] \to \operatorname{Aut}(\mathbf{D}^{\mathrm{b}}(A))^{\langle -1\rangle} \to \mathrm{Sp}'(A) \to 0, \tag{2.3.9}$$

and the group of invariant autoequivalences fits into the diagram

$$\operatorname{Aut}(\mathbf{D}^{\mathrm{b}}(A))^{\langle -1\rangle} \twoheadleftarrow \operatorname{Aut}(\mathbf{D}^{\mathrm{b}}(A))^{\mathrm{h}\langle -1\rangle} \to \operatorname{Aut}(\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^1(A))),$$

where both maps have kernels isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Proof. — See [Plo07, §3.2] or [Plo05, §4.2]. Sequence (2.3.9) arises by taking $\mathbb{Z}/2\mathbb{Z}$-invariants of Seq. (2.3.3) in Theorem 2.3.8. The right exactness of the sequence results from the vanishing of the group cohomology group

$$\mathrm{H}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z} \times A \times A^\vee) = 0.$$

Note that $\mathrm{Sp}'(A)$ is unaffected by taking invariants, since the action is given by conjugation with

$$\gamma_A([-1]_A^*) = \mathrm{diag}(-\mathrm{id}, -\mathrm{id}),$$

cf. Example 2.3.11.

The second diagram in the statement comes directly from Theorem 2.2.13 in view of the identification (2.3.8), the vanishing $\mathrm{H}^2(\mathbb{Z}/2\mathbb{Z}, \Bbbk^\times) = 0$, and the facts $\mathrm{Z}(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \Bbbk^\times) \simeq \mathbb{Z}/2\mathbb{Z}$. □

2.3.19. Remark. — In Part II we prove the analog of Theorem 2.3.18 for higher dimensional generalized Kummer varieties, cf. Theorem 5.2.4. In particular, we study the vanishing of group cohomology in degree 1 in this setting, cf. §4.2, and the invariant elements in the Mukai–Polishchuk group, cf. §5.1. We invite the reader to contrast the details of our results with Sequence (2.3.9) in Theorem 2.3.18.

We will also generalize the train of thought in the proof of Theorem 2.3.17 beyond the case $n = 2$, whereby we recover in particular the theorem just mentioned, cf. Remark 6.1.9.

CHAPTER 3

Group cohomology

3.1. Abelian group cohomology

There are many references for the theory of group cohomology, we will refer to [WeiIHA, Ch. 6], and [NSW, Ch. I], and [BroCG]. The way we define various notions is not meant to be the most efficient ansatz, but rather tries to, firstly, be robust regarding the different fundamental approaches to group cohomology and, secondly, provide some details that are otherwise often unstressed in the literature but useful for computations later on. Everything in this section is standard material, except for the proof of Proposition 3.1.23, which we could not locate in the literature; it allows us to provide a streamlined proof of Proposition 3.1.24.

3.1.1. Situation. — Let G be a group acting (from the left) on an abelian group A by homomorphisms.

3.1.2. — That is, A is a $\mathbb{Z}[G]$-module, where the *group algebra* $\mathbb{Z}[G]$ is the free abelian group generated by the elements of G and multiplication is extended linearly from the multiplication law of G. Thus, a morphism $\varphi\colon A \to B$ of $\mathbb{Z}[G]$-modules is a group homomorphism which is equivariant, i.e. $\varphi(g.a) = g.\varphi(b)$ for every $g \in G$ and $a \in A$. The homomorphism group $\mathrm{Hom}_{\mathbb{Z}}(A, B)$ becomes a $\mathbb{Z}[G]$-module via the action

$$g.\varphi \coloneqq g. \circ \varphi \circ g.^{-1},$$

so that its fixed points (cf. ¶3.1.3) are exactly the equivariant homomorphisms

$$\mathrm{Hom}_{\mathbb{Z}[G]}(A, B) = \mathrm{Hom}_{\mathbb{Z}}(A, B)^G.$$

3.1.3. — Define $A^G \coloneqq \{a \in A \mid g.a = a \text{ for all } g \in G\}$ as the subgroup of *G-fixed points* of A. The elements of A^G are also called *G-invariant*. The construction of taking G-invariants becomes a functor

$$(-)^G\colon \mathbf{Mod}(\mathbb{Z}[G]) \to \mathbf{Mod}(\mathbb{Z})$$

by sending morphism to their underlying group homomorphisms. Note that we have a natural isomorphism of functors

$$(-)^G \simeq \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -),$$

where $\mathbb{Z}$ carries the trivial G-action. We see that these are additive functors of abelian categories (with enough injectives), which are moreover left-exact.

Similarly, the G-coinvariants of A are $A_G \coloneqq A/\langle g.a - a \mid g \in G, a \in A\rangle$. Since $\mathbb{Z}[G]_G \simeq \mathbb{Z}$ with trivial action, we obtain a natural isomorphism of right-exact functors

$$(-)_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}[G]} (-)\colon \mathbf{Mod}(\mathbb{Z}[G]) \to \mathbf{Mod}(\mathbb{Z}).$$

***3.1.4. Definition* (Group cohomology).** — The *group cohomology* $\mathrm{H}^i(G, A)$ of G in degree i with values in A is the right derived functor of $(-)^G$ evaluated at A

$$\mathrm{H}^i(G, A) \coloneqq \mathbf{R}^i(A)^G \simeq \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

The *group homology* $\mathrm{H}_i(G, A)$ of G with values in A in degree i is the left derived functor of $(-)_G$ evaluated at A

$$\mathrm{H}_i(G, A) \coloneqq \mathbf{L}^i(A)_G \simeq \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

3.1.5. — From the description as derived functors, we have natural isomorphisms $\mathrm{H}^0(G, A) \simeq A^G$ and $\mathrm{H}_0(G, A) \simeq A_G$. We will treat these isomorphisms as equalities in the remainder of this text. For every short exact sequence $0 \to A' \to A \to A'' \to 0$ of $\mathbb{Z}[G]$-modules, we obtain a long exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^0(G, A') \to \mathrm{H}^0(G, A) \to \mathrm{H}^0(G, A'') \\ &\to \mathrm{H}^1(G, A') \to \mathrm{H}^1(G, A) \to \mathrm{H}^1(G, A'') \\ &\to \mathrm{H}^2(G, A') \to \dots \end{aligned}$$

and these sequences are natural with respect to morphisms of short exact sequences.

***3.1.6. Definition* (Crossed homomorphisms).** — A *crossed homomorphism* $\varphi\colon G \to A$ is a map satisfying

$$\varphi(gh) = \varphi(g) + g.\varphi(h) \tag{3.1.1}$$

for every $g, h \in G$. The map φ is a *principal crossed homomorphism* if there exists some $a_0 \in A$ such that $\varphi(g) = g.a_0 - a_0$ for all $g \in G$.

3.1.7. — A crossed homomorphism $\varphi\colon G \to A$ satisfies in particular $\varphi(1) = 0$ and $\varphi(g^{-1}) = -g^{-1}.\varphi(g)$. When the group G is given by a presentation $G = \langle g_1, \dots, g_n \mid r_1, \dots, r_m\rangle$, then a crossed homomorphism $\varphi\colon G \to A$ can be specified on the generators and extended to all of G by successive applications of (3.1.1) as long as these extensions satisfy $\varphi(r_i) = 1$ for all the relations r_i.

3.1.8. Example. — In the proof of Proposition 5.2.7 we will use the Coxeter–Moore presentation of the symmetric group. The symmetric group S_n is generated by the transpositions $\tau_i \coloneqq (i, i+1)$ for $i = 1, \dots, n-1$ subject to the relations

- $\tau_i^2 = 1$ for $i = 1, \dots, n-1$,
- $\tau_i\tau_j = \tau_j\tau_i$ for $|i - j| > 1$, and
- $(\tau_i\tau_{i+1})^3 = 1$ for $i = 1, \dots, n-2$,

which realize symmetric groups as Coxeter groups, cf. [BB, Ex. 1.2.3].

3.1.9. **Bar resolution.** — To facilitate computations, a more concrete description of group cohomology using cocycles is convenient. By the description of group cohomology using Ext groups, one can use a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module in computations. Following [WeiIHA, §6.5], the *unnormalized bar complex* of G is the complex $B_\bullet$ where B_n is the free $\mathbb{Z}[G]$-module generated by the symbols $[g_1, \dots, g_n]$ for $g_i \in G$, where the empty symbol $[\,]$ is valid, and $\mathrm{d}\colon B_n \to B_{n-1}$ is defined as

$$\mathrm{d}([g_1, \dots, g_n]) = g_1.[g_2, \dots, g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1, \dots, g_i g_{i+1}, \dots, g_n] + (-1)^n [g_1, \dots, g_{n-1}],$$

see (3.1.2)–(3.1.4) for concrete examples of this formula. Then, by augmenting the bar complex by $B_0 = \mathbb{Z}[G] \to \mathbb{Z}$ with $g \mapsto 1$, the complex $B_\bullet \to \mathbb{Z} \to 0$ becomes a free resolution of the $\mathbb{Z}[G]$-module $\mathbb{Z}$, cf. [WeiIHA, Thm. 6.5.3]. So

$$\mathrm{H}^i(G, A) \simeq \mathrm{H}^i(\mathrm{Hom}_{\mathbb{Z}[G]}(B_\bullet, A)) \quad \text{and} \quad \mathrm{H}_i(G, A) \simeq \mathrm{H}_i(B_\bullet \otimes_{\mathbb{Z}[G]} A)$$

by their description as Ext- and Tor-groups, respectively. In this viewpoint, we call elements of $\mathrm{H}^i(G, A)$ *cocycle classes*; elements in the kernel of the differentials of $\mathrm{Hom}_{\mathbb{Z}[G]}(B_\bullet, A)$ are called *cocycles* and elements in their image are called *coboundaries*. For example, we have the following formulas of low degree differentials

$$\mathrm{d}([h]) = h.[\,] - [\,], \tag{3.1.2}$$

$$\mathrm{d}([g, h]) = g.[h] - [gh] + [g], \tag{3.1.3}$$

$$\mathrm{d}([f, g, h]) = f.[g, h] - [fg, h] + [f, gh] - [f, g]. \tag{3.1.4}$$

In particular, $\mathrm{H}^1(G, A)$ consists of classes of crossed homomorphisms modulo principal crossed homomorphisms. Hence, if G acts trivially on A, then

$$\mathrm{H}^1(G, A) \simeq \mathrm{Hom}(G, A) \simeq \mathrm{Hom}(G^{\mathrm{ab}}, A), \tag{3.1.5}$$

and when G is finite,

$$\mathrm{H}^1(G, \mathbb{Z}) \simeq \mathrm{Hom}(G, \mathbb{Z}) = 0$$

since $\mathbb{Z}$ is torsion free.

For group homology we obtain that $\mathrm{H}_1(G, \mathbb{Z})$ is the free abelian group generated by the symbols $[g]$ for $g \in G$ modulo the relations $[h] - [gh] + [g] = 0$ for $g, h \in G$. More generally when G acts trivially on A, we have

$$\mathrm{H}_1(G, A) \simeq G^{\mathrm{ab}} \otimes_{\mathbb{Z}} A. \tag{3.1.6}$$

3.1.10. Remark. — The descriptions of first group homology and cohomology with trivial coefficients can also be proven without the bar complex by considering the *augmentation ideal* and the universal coefficient theorem, cf. [WeiIHA, Thm. 6.1.12, Cor. 6.4.6].

From a more abstract point of view, $\mathrm{H}^2(G, A)$ consists of isomorphism classes of group extensions of G by A such that the conjugation action of G on A coincides with the given $\mathbb{Z}[G]$-module structure on A, cf. [WeiIHA, Thm. 6.6.3]. The group $\mathrm{H}^1(G, A)$ consists of isomorphism classes of G-*equivariant* A-*torsors*, see §3.2 for details.

3.1.11. Remark. — There is a homotopical approach to group cohomology which motivates some of its good properties, like the universal coefficient theorem for example. See [AM, Ch. II] and [WeiIHA, §6.10] for details. The *classifying space* $\mathrm{B}G$ of the discrete group G is an Eilenberg–MacLane space $\mathrm{K}(G,1)$, i.e. it is a connected topological space with

$$\pi_1(\mathrm{B}G, \mathrm{pt}) \simeq G$$

and $\pi_i(\mathrm{B}G,\mathrm{pt}) = 0$ for $i \neq 1$. It can be constructed as the quotient $\mathrm{B}G = \mathrm{E}G/G$ of a contractible space $\mathrm{E}G$ on which G acts freely. Note that a $\mathbb{Z}[G]$-module A corresponds to a local system $\underline{A}$ on $\mathrm{B}G$ since the latter are nothing else than $\mathbb{Z}[\pi_1(\mathrm{B}G,\mathrm{pt})]$-modules. There is a particular model of $\mathrm{E}G$ whose cellular chain complex $\mathrm{C}^{\mathrm{cell}}_\bullet(\mathrm{E}G)$ is exactly the bar complex of the group G, cf. [AM, Ex. II.3.4]. Thus the group cohomology of G with values in A coincides with the singular cohomology with local coefficients

$$\mathrm{H}^i_{\mathrm{sing}}(\mathrm{B}G, \underline{A}) \simeq \mathrm{H}^i(\mathrm{Hom}_{\mathbb{Z}[\pi_1(\mathrm{B}G,\mathrm{pt})]}(\mathrm{C}^{\mathrm{cell}}_\bullet(\mathrm{E}G), A)) \simeq \mathrm{H}^i(G, A).$$

***3.1.12. Proposition* (Universal coefficient theorem).** — *Let G be a group, let A a $\mathbb{Z}[G]$-module, and let B a $\mathbb{Z}$-module. Assume that A is free as a $\mathbb{Z}$-module, and that $i \geq 0$. Then there exists (non-canonically) split short exact sequences*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{i-1}(G,A), B) \to \mathrm{H}^i(G, \mathrm{Hom}_{\mathbb{Z}}(A,B)) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_i(G,A), B) \to 0,$$

and

$$0 \to \mathrm{H}_i(G,A) \otimes_{\mathbb{Z}} B \to \mathrm{H}_i(G, A \otimes_{\mathbb{Z}} B) \to \mathrm{Tor}^1_{\mathbb{Z}}(\mathrm{H}_{i-1}(G,A), B) \to 0.$$

Moreover the sequences are functorial with respect to A, B, and G.

Proof. — Let $P_\bullet \to \mathbb{Z}$ be a resolution of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules. Then all $P_j \otimes_{\mathbb{Z}[G]} A$ are free $\mathbb{Z}$-modules since A is free over $\mathbb{Z}$. Recall that submodules of free $\mathbb{Z}$-modules are themselves free. Now [WeiIHA, Thm. 3.6.5] provides the split short exact sequence

$$\begin{aligned}0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{i-1}(P_\bullet \otimes_{\mathbb{Z}[G]} A), B) \to \mathrm{H}^i(\mathrm{Hom}_{\mathbb{Z}}(P_\bullet \otimes_{\mathbb{Z}[G]} A, B))& \\ \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_i(P_\bullet \otimes_{\mathbb{Z}[G]} A), B) \to 0.&\end{aligned}$$

Note that for the outer terms

$$\mathrm{H}_{i-1}(P_\bullet \otimes_{\mathbb{Z}[G]} A) \simeq \mathrm{Tor}^{\mathbb{Z}}_{i-1}(\mathbb{Z}, A) \simeq \mathrm{H}_{i-1}(G, A),$$

and for the middle term

$$\mathrm{Hom}_{\mathbb{Z}}(P_\bullet \otimes_{\mathbb{Z}[G]} A, B) \simeq \mathrm{Hom}_{\mathbb{Z}[G]}(P_\bullet, \mathrm{Hom}_{\mathbb{Z}}(A,B))$$

by tensor-hom adjunction. The cohomology of the latter is group cohomology by its description via Ext groups.

The proof of the second claimed short exact sequence follows the same strategy, but using [WeiIHA, Thm. 3.6.1, Thm. 3.6.2] this time.

Regarding functoriality of the sequences. Let $f\colon H \to G$ be a group homomorphism, which induces a homomorphism $f\colon \mathbb{Z}[H] \to \mathbb{Z}[G]$. Take a free resolution $P_\bullet \to \mathbb{Z}$ of $\mathbb{Z}[G]$-modules and a free resolution $Q_\bullet \to \mathbb{Z}$ of $\mathbb{Z}[H]$-modules. Then $f_*P_\bullet \to \mathbb{Z}$ is still a resolution of $\mathbb{Z}[H]$-modules, since f_* is exact. Since the Q_i are free and in particular projective, the map $\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}$ lifts to a morphism $Q_\bullet \to f_*P_\bullet$ of complexes of $\mathbb{Z}[H]$-modules. This induces a morphism of complexes

$$Q_\bullet \otimes_{\mathbb{Z}[H]} A \to P_\bullet \otimes_{\mathbb{Z}[G]} A.$$

Finally, the constructions in [WeiIHA, §3.6] are functorial, essentially since they arise through long exact cohomology sequences associated to certain short exact sequences. □

3.1.13. Remark. — We can factor the map $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(G,\mathbb{Z}),B) \to \mathrm{H}^2(G,B)$ in the universal coefficient theorem as

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(G,\mathbb{Z}),B) \simeq \mathrm{Ext}^1_{\mathbb{Z}}(G^{\mathrm{ab}},B) \xrightarrow{\alpha} \mathrm{H}^2(G^{\mathrm{ab}},B) \xrightarrow{\beta} \mathrm{H}^2(G,B)$$

and this is compatible with viewing elements of these groups as group extensions, more precisely, α maps an extension of the abelian groups to the very same extensions viewed as a central extension, and β pulls a central extension back along the abelianization morphism $G \to G^{\mathrm{ab}}$, cf. [Bey82, Thm. 1.8].

***3.1.14.* Restricted representations.** — Let $f\colon H \to G$ be a homomorphism of groups, and let A be a $\mathbb{Z}[G]$-module. Define the $\mathbb{Z}[H]$-module as

$$\mathrm{Res}_f(A) := A$$

where the group H acts via f through the action of G on A. This construction becomes a functor

$$\mathrm{Res}_f\colon \mathbf{Mod}(\mathbb{Z}[G]) \to \mathbf{Mod}(\mathbb{Z}[H])$$

by sending a homomorphism of $\mathbb{Z}[G]$-modules to itself. It is clear that Res_f is exact, since it does not modify underlying abelian groups in any way. In the case that $f\colon H \hookrightarrow G$ is a subgroup, we call

$$\mathrm{Res}^G_H := \mathrm{Res}_f$$

the *restriction functor*.

3.1.15. — By construction of group cohomology as a derived functor, it is clear that $\mathrm{H}^\bullet(G,A)$ is functorial in $A \in \mathbf{Mod}(\mathbb{Z}[G])$. For a homomorphism $\varphi\colon A \to B$ of $\mathbb{Z}[G]$-modules we write $\varphi_*\colon \mathrm{H}^n(G,A) \to \mathrm{H}^n(G,B)$ for the induced functors.

Let $f\colon H \to G$ be a homomorphism of groups, and let A be a $\mathbb{Z}[G]$-module. We have a natural inclusion $A^G \hookrightarrow A^H = \mathrm{Res}_f(A)^H$ which extends uniquely to a morphism of δ-functors

$$\mathrm{res}_f\colon \mathrm{H}^\bullet(G,-) \to \mathrm{H}^\bullet(H,\mathrm{Res}_f(-)),$$

since the left hand side is a universal δ-functor by construction, and the right hand side is similarly a δ-functor in view of the exactness of Res_f. In the case that $f\colon H \hookrightarrow G$ is a subgroup, we write

$$\mathrm{res}^G_H\colon \mathrm{H}^\bullet(G,-) \to \mathrm{H}^\bullet(H,\mathrm{Res}^G_H(-)).$$

In more down-to-earth terms, representing an element of $\mathrm{H}^i(G,A)$ by a cocycle $\varphi\colon G^{\times i} \to A$, we obtain $\mathrm{res}^G_H([\varphi]) \in \mathrm{H}^i(H,\mathrm{Res}^G_H(A))$ by composing φ with the map $H^{\times i} \to G^{\times i}$ induced by f.

***3.1.16.* Induced representations.** — Let $H \subset G$ be a subgroup, so $\mathbb{Z}[G]$ becomes a (left and right) $\mathbb{Z}[H]$-module, and let B be a $\mathbb{Z}[H]$-module. The *coinduced* $\mathbb{Z}[G]$-module is defined as

$$\begin{aligned}\operatorname{CoInd}_H^G(B) &:= \operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], B) \\ &\simeq \{\phi\colon G \to B \mid \phi(hg) = h.\phi(g) \quad \text{for } g \in G, h \in H\}\end{aligned}$$

with G-action afforded by $g.\phi := (g' \mapsto \phi(g'g))$. The *induced* $\mathbb{Z}[G]$-module is defined as

$$\operatorname{Ind}_H^G(B) := \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} B.$$

The canonical projection

$$\pi\colon \operatorname{CoInd}_H^G(B) \to B, \quad \phi \mapsto \phi(1)$$

is a morphism of $\mathbb{Z}[H]$-modules. Then the functor $\operatorname{Ind}_H^G(-)$ is left adjoint to the restriction functor Res_H^G, while $\operatorname{CoInd}_H^G(-)$ is its right adjoint, cf. Frobenius reciprocity [BroCG, III.§3, III.§5].

If $H \subset G$ is a subgroup of finite index, we have an isomorphism of functors

$$\operatorname{Ind}_H^G(-) \simeq \operatorname{CoInd}_H^G(-), \tag{3.1.7}$$

cf. [WeiIHA, Lem. 6.3.4]. For this reason we won't make in this case a distinction in our terminology regarding "induced" and "coinduced" and just use the notation $\operatorname{Ind}_H^G(B)$ for both.

The following perspective on induced representations is useful when considering a $\mathbb{Z}[H]$-module with trivial action, and is applied later in the proof of Proposition 4.2.4.

3.1.17. — Assume $H \subset G$ is a subgroup of finite index, and let $g_1, \ldots, g_n$ be a set of right coset representatives for $H\backslash G$. Then we have an isomorphism

$$\operatorname{CoInd}_H^G(B) \simeq \operatorname{Map}(H\backslash G, B),$$

sending ϕ to the map of sets $Hg_i \mapsto \phi(g_i)$. Its inverse is given by extending such a map of sets via the formula $\phi(hg) = h.\phi(g)$. The left G-action on $\operatorname{Map}(H\backslash G, B)$ becomes

$$(g.\phi)(Hg_i) := h.\phi(Hg_j),$$

when $g_i g = hg_j$ for $g \in G$ and $h \in H$.

Note that from this point of view, an isomorphism $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} B \xrightarrow{\sim} \operatorname{Map}(H\backslash G, B)$ as in (3.1.7) is given by mapping $g_i^{-1}h \otimes b \mapsto h.b \cdot \delta_{Hg_i}$, where δ_{Hg_i} denotes the Kronecker symbol.

Assume now that B arises as the restriction of a $\mathbb{Z}[G]$-module with trivial action. Note that then the isomorphism above is independent of the choice of coset representatives $\{g_i\}$. The G-action on $\operatorname{Map}(H\backslash G, B)$ above becomes just the usual action on the set of maps between a right and a left G-set. Concretely, for $\phi \in \operatorname{Map}(H\backslash G, B)$ and $g \in G$ this is given by

$$\begin{aligned} g.\phi &= x \mapsto g.\phi(x.g) \\ &= x \mapsto \phi(x.g). \end{aligned}$$

3.1.18. Proposition **(Shapiro's Lemma).** — *Let G be a group with some subgroup $H \subset G$, and let B be a $\mathbb{Z}[H]$-module. Then, for every $n \geq 0$, the canonical projection π induces an isomorphism*

$$\mathrm{sh}_n\colon \mathrm{H}^n(G, \mathrm{CoInd}_H^G(B)) \xrightarrow{\sim} \mathrm{H}^n(H, B). \tag{3.1.8}$$

Proof. — See [NSW, Prop. 1.6.4] or [WeiIHA, Lem. 6.3.2]. We provide a sketch using δ-functors. It is straightforward to check that $\pi_* \circ \mathrm{res}_H^G$ is an isomorphism in degree 0. We know that the right hand side of (3.1.8) is a universal δ-functor, and the left hand side is a δ-functor since CoInd_H^G is exact by the freeness of $\mathbb{Z}[G]$ as a $\mathbb{Z}[H]$-module (with basis given by a set of coset representatives). The left hand side is a universal δ-functor, since it is erasable for $n \geq 1$. Indeed, coinduction preserves injective objects, since it is right adjoint to the restriction functor which preserves monomorphisms. So the morphism

$$\mathrm{sh}_n \coloneqq \pi_* \circ \mathrm{res}_H^G\colon \mathrm{H}^n(G, \mathrm{CoInd}_H^G(B)) \xrightarrow{\sim} \mathrm{H}^n(H, B)$$

becomes an isomorphism as it is one in degree 0. □

3.1.19. Situation. — From now on we assume that G is a group and $H \subset G$ is a subgroup of finite index. In particular we won't make a distinction between the induction and coinduction functors any more.

3.1.20. Definition. — Let $H \subset G$ be a subgroup of finite index, and let A be a $\mathbb{Z}[G]$-module. Define the following two homomorphisms of $\mathbb{Z}[G]$-modules:

$$\iota\colon A \to \mathrm{Ind}_H^G(\mathrm{Res}_H^G(A)) \qquad x \mapsto (g \mapsto g.x) \tag{3.1.9}$$

$$\nu\colon \mathrm{Ind}_H^G(\mathrm{Res}_H^G(A)) \to A \qquad \phi \mapsto \sum_{[g] \in G/H} g.\phi(g^{-1}). \tag{3.1.10}$$

3.1.21. Remark. — The significance of the maps in Definition 3.1.20 is that ι is the unit of the restriction-coinduction adjunction, and ν is the counit of the induction-restriction adjunction, cf. [NSW, p. 63]. Also, the map ι is injective and ν is surjective, in accord with the faithfulness of the restriction functor.

3.1.22. **Corestriction.** — Besides the contravariant functoriality of $\mathrm{H}^\bullet(G, A)$ in G given by restriction, group cohomology is also covariantly functorial with respect to finite index subgroups $H \subset G$ via *corestriction* maps (also called *transfer* maps)

$$\mathrm{cor}_H^G\colon \mathrm{H}^\bullet(H, \mathrm{Res}_H^G(A)) \to \mathrm{H}^\bullet(G, A).$$

In degree 0, the corestriction map is the norm map

$$\mathrm{cor}_H^G\colon A^H \to A^G, \quad a \mapsto \sum_{[g] \in G/H} g.a. \tag{3.1.11}$$

This norm map extends then uniquely to a morphisms of universal δ-functors, cf. Proposition 3.1.23.

3.1.23. Proposition. — *Let $H \subset G$ be a subgroup which is of finite index or normal. Then the functors $\mathrm{H}^n(H, \mathrm{Res}^G_H(-))\colon \mathbf{Mod}(\mathbb{Z}[G]) \to \mathbf{Mod}(\mathbb{Z})$ form a universal δ-functor.*

Proof. — It is clear that $\mathrm{H}^\bullet(H, \mathrm{Res}^G_H(-))$ is a δ-functor, since $\mathrm{Res}^G_H(-)$ is an exact functor. Regarding universality, we check that that it is an erasable functor, i.e. for every $\mathbb{Z}[G]$-module A, there exists an injection $A \hookrightarrow \widetilde{A}$ of $\mathbb{Z}[G]$-modules such that $\mathrm{H}^{\geq 1}(H, \mathrm{Res}^G_H(\widetilde{A})) = 0$. By Definition 3.1.20 and Remark 3.1.21 we have an injection

$$A \hookrightarrow \mathrm{Ind}^G_0(\mathrm{Res}^G_0(A)) =: \widetilde{A}.$$

Let us abbreviate $A_0 := \mathrm{Res}^G_0(A)$. By induction on the index $(G : H)$, we check that

$$\mathrm{H}^{\geq 1}(H, \mathrm{Res}^G_H(\widetilde{A})) = 0.$$

The case $(G : H) = 1$ is trivial since then $H = G$, and $\mathrm{Res}^G_H = \mathrm{id}$ and by Shapiro's lemma (Proposition 3.1.18)

$$\mathrm{H}^{\geq 1}(G, \mathrm{Ind}^G_0(A_0)) \simeq \mathrm{H}^{\geq 1}(0, A_0) = 0.$$

Otherwise we use $\mathrm{Ind}^G_0(A_0) \simeq \mathrm{Ind}^G_H(\mathrm{Ind}^H_0(A_0))$, abbreviate $\widetilde{A}' := \mathrm{Ind}^G_0(A_0)$, and use Mackey's formula

$$\mathrm{Res}^G_H(\mathrm{Ind}^G_H(\widetilde{A}')) \simeq \bigoplus_{[\sigma] \in H\backslash G/H} \mathrm{Ind}^H_{H\cap\sigma H\sigma^{-1}}({}^{\sigma}\,\mathrm{Res}^H_{H\cap\sigma^{-1}H\sigma}(\widetilde{A}')),$$

cf. [BroCG, Prop. III.5.6.b]. Now we get for each of the direct summands that

$$\begin{aligned} &\mathrm{H}^{\geq 1}(H, \mathrm{Ind}^H_{H\cap\sigma H\sigma^{-1}}({}^{\sigma}\,\mathrm{Res}^H_{H\cap\sigma^{-1}H\sigma}(\widetilde{A}'))) \\ \simeq\, &\mathrm{H}^{\geq 1}(H \cap \sigma H\sigma^{-1}, {}^{\sigma}\,\mathrm{Res}^H_{H\cap\sigma^{-1}H\sigma}(\widetilde{A}')) \\ \simeq\, &\mathrm{H}^{\geq 1}(\sigma^{-1}H\sigma \cap H, \mathrm{Res}^H_{H\cap\sigma^{-1}H\sigma}(\widetilde{A}')), \end{aligned}$$

where the first isomorphism is Shapiro's lemma (Proposition 3.1.18), and the second isomorphism comes from conjugation via $H \cap \sigma H\sigma^{-1} \xrightarrow{\sim} \sigma^{-1}H\sigma \cap H$, $h \mapsto \sigma^{-1}h\sigma$. The latter cohomology group vanishes by induction hypothesis. Indeed, by the index of an intersection formula, cf. [CohAlg, §3.3, Exer. 9], we have

$$\big(G : \sigma^{-1}H\sigma \cap H\big) \leq (G : H)\,\big(G : \sigma^{-1}H\sigma\big)\,,$$

which implies

$$\big(H : \sigma^{-1}H\sigma\big) \leq \big(G : \sigma^{-1}H\sigma\big) = (G : H)$$

and we have equality if and only if $G = \sigma^{-1}H\sigma H$. The latter is equivalent to $G = H\sigma H$ and would mean that $G = H$ and there is only one double coset.

In the case that $H \subset G$ is normal, Mackey's formula simplifies so that an application of Shapiro's lemma suffices to finish the proof without using induction. □

The next proposition could be used to give an alternative definition of corestriction, but, for the sake of robustness, we will not take this shortcut.

3.1.24. Proposition. — *Let G be a finite group with some subgroup $H \subset G$, and let A be a $\mathbb{Z}[G]$-module. For $n \geq 0$ we have commutative diagrams*

$$\begin{tikzcd} \mathrm{H}^n(G,A) \arrow[r, "\iota_*"] \arrow[rr, bend right, "\mathrm{res}^G_H"'] & \mathrm{H}^n(G, \mathrm{Ind}^G_H(\mathrm{Res}^G_H(A))) \arrow[r, "\mathrm{sh}"] & \mathrm{H}^n(H, \mathrm{Res}^G_H(A)), \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{H}^n(H, \mathrm{Res}^G_H(A)) \arrow[r, "\mathrm{sh}^{-1}"] \arrow[rr, bend right, "\mathrm{cor}^G_H"'] & \mathrm{H}^n(G, \mathrm{Ind}^G_H(\mathrm{Res}^G_H(A))) \arrow[r, "\nu_*"] & \mathrm{H}^n(G,A). \end{tikzcd}$$

Proof. — See [NSW, Prop. 1.6.5] for a concrete proof on the level of cocycles. We provide a proof using δ-functors. First we check the identities on degree 0. The Shapiro isomorphism sh is just the map

$$\pi\colon \mathrm{Ind}^G_H(\mathrm{Res}^G_H(A))^G \to \mathrm{Res}^G_H(A)^H, \quad \phi \mapsto \phi(1).$$

The push-forward ι_* becomes

$$\iota\colon A^G \to \mathrm{Ind}^G_H(\mathrm{Res}^G_H(A))^G, \quad x \mapsto (g \mapsto g.x).$$

Their composition is the inclusion $N^G \hookrightarrow N^H$, which is exactly the restriction map.

Next, recall that the corestriction map in degree 0 is the norm map (3.1.11), and $\mathrm{sh}(\phi) = \phi(1)$ as above. For $\phi \in \mathrm{H}^0(G, \mathrm{Ind}^G_H(\mathrm{Res}^G_H(A)))$ we have by definition

$$\nu(\phi) = \sum_{[g]\in G/H} g.\phi(g^{-1}) = \sum_{[g]\in G/H} g.\phi(1),$$

where we used that for every $g \in G$ we have $g.\phi = \phi$, which implies $\phi(g) = \phi(1)$. In conclusion, the maps agree, as desired.

We conclude using the fact that res^G_H and $\mathrm{sh} \circ \iota_*$, as well as $\nu_* \circ \mathrm{sh}^{-1}$ and cor^G_H are morphisms of universal δ-functors. □

3.1.25. Proposition. — *Let $H \subset G$ be subgroup of finite index, and let $n \geq 0$. Then we have for every $\mathbb{Z}[G]$-module A the identity of endomorphisms of $\mathrm{H}^n(G,A)$*

$$\mathrm{cor}^G_H \circ \mathrm{res}^G_H = (G : H) \cdot \mathrm{id}\,.$$

Proof. — See [WeiIHA, Lem. 6.7.17]. It is enough to check the identity in degree 0, since we are comparing morphisms of universal δ-functors. For $n = 0$ and $a \in A^G$ we have by definition that

$$(\mathrm{cor}^G_H \circ \mathrm{res}^G_H)(a) = \sum_{[g]\in G/H} g.a = \sum_{[g]\in G/H} a = (G : H)\, a. \qquad \square$$

3.1.26. Remark. — Proposition 3.1.25 instantiated with a finite group G and $H = 0$ implies that that multiplication by the order $\#G$ of G is the zero map on the cohomology groups $\mathrm{H}^n(G,A)$ for $n \geq 1$. Also note that whenever A is finitely generated as an abelian group, then all $\mathrm{H}^n(G,A)$ are finitely generated as well, which is clear from the description in ¶3.1.9. In particular, the higher cohomology groups, $n \geq 1$, are finite abelian groups in this case.

***3.1.27. Definition* (Schur multiplier).** — The *Schur multiplier* of a group G is the group homology group $\mathrm{H}_2(G,\mathbb{Z})$, where G acts trivially on $\mathbb{Z}$.

3.1.28. Notation. — We allow, by abuse of notation, to refer to $\mathrm{H}^2(G,\Bbbk^\times)$ also by "Schur multiplier" when $\Bbbk$ is an algebraically closed field of characteristic zero. This is somewhat justified by Proposition 3.1.29.(ii) below.

3.1.29. Proposition. — *Assume that G is a finite group.*

(i) *Then $\mathrm{H}_2(G,\mathbb{Z})$ is a finite abelian group, consisting of $\#G$-torsion elements.*

(ii) *Let $\Bbbk$ be an algebraically closed field of characteristic zero. Considering $\Bbbk^\times$ endowed with trivial G-action, we have*

$$\mathrm{H}^n(G,\Bbbk^\times)\simeq \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(G,\mathbb{Z}),\Bbbk^\times).$$

In particular, the Schur multiplier $\mathrm{H}_2(G,\mathbb{Z})$ and $\mathrm{H}^2(G,\Bbbk^\times)$ are isomorphic as abstract groups.

Proof. — (i) Since $\mathbb{Z}$ is finitely generated, $\mathrm{H}_n(G,\mathbb{Z})$ is a finitely generated abelian group as well. For $n\geq 1$, Remark 3.1.26 tells us that $\mathrm{H}^n(G,\mathbb{Z})$ is finite abelian group which is annihilated by multiplication by $\#G$. The universal coefficient theorem (Proposition 3.1.12) for group cohomology, instantiated with $A=\mathbb{Z}$ and $B=\mathbb{Z}$, implies that $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(G,\mathbb{Z}),\mathbb{Z})$ is also a finite group, hence necessarily zero. We conclude that $\mathrm{H}_n(G,\mathbb{Z})$ is finite since its rank must be zero. The second part of the claim follows from (ii) since $\mathrm{H}^n(G,\Bbbk^\times)$ consists of $\#G$-torsion elements by Remark 3.1.26.

(ii) By the universal coefficient theorem (Proposition 3.1.12) we have a short exact sequence

$$0\to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{n-1}(G,\mathbb{Z}),\Bbbk^\times)\to \mathrm{H}^n(G,\Bbbk^\times)\to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(G,\mathbb{Z}),\Bbbk^\times)\to 0.$$

We have $\mathrm{H}_0(G,\mathbb{Z})=\mathbb{Z}$, and in the proof of (i) we saw that $\mathrm{H}_{n-1}(G,\mathbb{Z})$ is a finite abelian group for $n\geq 2$, so it is isomorphic to a direct sum of finite cyclic groups. Now, use that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z},\Bbbk^\times)\simeq \Bbbk^\times/(\Bbbk^\times)^m=1$ as well as $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\Bbbk^\times)=1$.

For the second claim in (ii), write again $\mathrm{H}_2(G,\mathbb{Z})$ as a direct sum of finite cyclic groups. Now note that $\mathbb{Z}/m\mathbb{Z}$ is isomorphic to the group of roots of unity $\mu_m(\Bbbk)\simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z},\Bbbk^\times)$. □

3.1.30. Example. — Let us list a few examples of Schur multipliers, where the one of the symmetric group will be of most importance to us later; its calculation goes back to Schur [Sch11].

Group G	$\mathrm{H}_2(G,\mathbb{Z})$	Condition	Reference
cyclic group $\mathbb{Z}/n\mathbb{Z}$	0	$n\in\mathbb{N}$	[KarSN, Thm. 2.1.1]
symmetric group S_n	0	$n\leq 3$	[KarSN, Thm. 2.12.3]
	$\mathbb{Z}/2\mathbb{Z}$	$n\geq 4$	
alternating group A_n	0	$n\leq 3$	[KarSN, Thm. 2.12.5]
	$\mathbb{Z}/6\mathbb{Z}$	$n=6,7$	
	$\mathbb{Z}/2\mathbb{Z}$	$n\geq 4$, $n\neq 6,7$	

TABLE 1. Group cohomology of symmetric groups with integer coefficients.

n	$\mathrm{H}^0(\mathrm{S}_n,\mathbb{Z})$	$\mathrm{H}^1(\mathrm{S}_n,\mathbb{Z})$	$\mathrm{H}^2(\mathrm{S}_n,\mathbb{Z})$	$\mathrm{H}^3(\mathrm{S}_n,\mathbb{Z})$	$\mathrm{H}^4(\mathrm{S}_n,\mathbb{Z})$	$\mathrm{H}^5(\mathrm{S}_n,\mathbb{Z})$	$\mathrm{H}^6(\mathrm{S}_n,\mathbb{Z})$
1	$\mathbb{Z}$	0	0	0	0	0	0
2	$\mathbb{Z}$	0	$\mathbb{F}_2$	0	$\mathbb{F}_2$	0	$\mathbb{F}_2^{\oplus 1}$
3	$\mathbb{Z}$	0	$\mathbb{F}_2$	0	$\mathbb{F}_2^{\oplus 1}\oplus\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{F}_2^{\oplus 1}$
4	$\mathbb{Z}$	0	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2^{\oplus 1}\oplus\mathbb{Z}/12\mathbb{Z}$	$\mathbb{F}_2^{\oplus 1}$	$\mathbb{F}_2^{\oplus 3}$
5	$\mathbb{Z}$	0	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2^{\oplus 1}\oplus\mathbb{Z}/12\mathbb{Z}$	$\mathbb{F}_2^{\oplus 1}$	$\mathbb{F}_2^{\oplus 3}$
6	$\mathbb{Z}$	0	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2^{\oplus 2}\oplus\mathbb{Z}/12\mathbb{Z}$	$\mathbb{F}_2^{\oplus 2}$	$\mathbb{F}_2^{\oplus 5}$
7	$\mathbb{Z}$	0	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2^{\oplus 2}\oplus\mathbb{Z}/12\mathbb{Z}$	$\mathbb{F}_2^{\oplus 2}$	$\mathbb{F}_2^{\oplus 5}$
8	$\mathbb{Z}$	0	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2^{\oplus 2}\oplus\mathbb{Z}/12\mathbb{Z}$	$\mathbb{F}_2^{\oplus 3}$	$\mathbb{F}_2^{\oplus 6}$
9	$\mathbb{Z}$	0	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2^{\oplus 2}\oplus\mathbb{Z}/12\mathbb{Z}$	$\mathbb{F}_2^{\oplus 3}$	$\mathbb{F}_2^{\oplus 6}$
10	$\mathbb{Z}$	0	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2^{\oplus 2}\oplus\mathbb{Z}/12\mathbb{Z}$	$\mathbb{F}_2^{\oplus 3}$	$\mathbb{F}_2^{\oplus 7}$
11	$\mathbb{Z}$	0	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2^{\oplus 2}\oplus\mathbb{Z}/12\mathbb{Z}$	$\mathbb{F}_2^{\oplus 3}$	$\mathbb{F}_2^{\oplus 7}$
12	$\mathbb{Z}$	0	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2^{\oplus 2}\oplus\mathbb{Z}/12\mathbb{Z}$	$\mathbb{F}_2^{\oplus 3}$	$\mathbb{F}_2^{\oplus 7}$

3.1.31. — Let us summarize the group cohomology of symmetric groups briefly. First, let us get out of the way that by (3.1.6)

$$\mathrm{H}_1(\mathrm{S}_n,\mathbb{Z}) \simeq \mathrm{S}_n^{\mathrm{ab}} = \mathrm{S}_n/\mathrm{A}_n \simeq \mathbb{Z}/2\mathbb{Z}$$

for $n \geq 2$; and for $n \geq 3$ the center is

$$\mathrm{Z}(\mathrm{S}_n) = \{\mathrm{id}\}.$$

See Table 1 for the low degree group cohomology of the symmetric group S_n with $n \leq 12$, which we have produced using the computer algebra system GAP [GAP] with the command `GroupCohomology(SymmetricGroup(n),k);` of the package `hap` [GAPhap]. For readability of the table we have used the notation $\mathbb{F}_2$ for the group $\mathbb{Z}/2\mathbb{Z}$.

The table suggests a stability result. Indeed, we have the following result (Theorem 3.1.32) due to Nakaoka. We have shaded the unstable range, according to this proposition, in gray. With integer coefficients, the table suggest a better unstability range, shaded in darker gray, but for example $\mathrm{H}^2(\mathrm{S}_4,\mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ is not isomorphic to $\mathrm{H}^2(\mathrm{S}_3,\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

***3.1.32. Theorem* (Nakaoka).** — *Let A be an abelian group endowed with trivial S_n-action. Then the restriction map*

$$\mathrm{res}^{\mathrm{S}_n}_{\mathrm{S}_{n-1}} \colon \mathrm{H}^k(\mathrm{S}_n, A) \to \mathrm{H}^k(\mathrm{S}_{n-1}, \mathrm{Res}^{\mathrm{S}_n}_{\mathrm{S}_{n-1}}(A))$$

is an isomorphism for $k < n/2$; it is always surjective.

Proof. — See [Nak60, Cor. 6.7, Thm. 5.8]. The lecture notes [Kup21] provide an exposition of Quillen's strategy for the result, cf. [Qui74; RW17; SW20]. □

3.2. Non-abelian group cohomology and equivariant torsors

We will consider in Part II group cohomology with values in a not-necessarily abelian group. Let us briefly collect in this section the results from non-abelian group cohomology which we will need later. We use [BerIGH, Ch. II] as a reference; alternatively, see [BS64, §1], [SerGC, §I.5] or [GirCNA, §III.3]. Besides other things, we discuss the notion of equivariant pseudo-torsors, which is not new but appears to be not very well known among algebraic geometers. Since we rely crucially on this notion later on, we spell out some ultimately elementary computations in detail.

3.2.1. Situation. — Let G be a group, and let A be a G-*group*, i.e. a group, written multiplicatively when it is not assumed to be abelian, endowed with a left action by G via group homomorphisms.

***3.2.2.* Non-abelian group cohomology sets.** — As before, define group cohomology in degree 0 as the subgroup

$$\mathrm{H}^0(G, A) := A^G$$

of A of G-invariant elements. The first non-abelian group cohomology set $\mathrm{H}^1(G, A)$ consists of equivalence classes of 1-cocycles. Recall that a 1-cocycle is nothing else than a crossed homomorphism $\varphi\colon G \to A$, i.e. a map satisfying

$$\varphi(gh) = \varphi(g)(g.\varphi(h))$$

for $g, h \in G$. The *trivial* 1*-cocycle* is the constant crossed homomorphism $\varphi_{\mathrm{triv}}\colon g \mapsto 1$. The group A acts from the left on the set of crossed homomorphism via

$$a.\varphi := g \mapsto a\varphi(g)(g.a^{-1}),$$

and we define

$$\mathrm{H}^1(G, A) := \mathrm{Hom}_{\mathrm{crossed}}(G, A)/A,$$

which is a pointed set with base-point φ_{triv}.

3.2.3. — Given a morphism $f\colon A \to B$ of G-groups, i.e. a G-equivariant homomorphism, we obtain by restriction an induced map $f_*\colon A^G \to B^G$, and by postcomposition an induced map $\mathrm{Hom}_{\mathrm{crossed}}(G, A) \to \mathrm{Hom}_{\mathrm{crossed}}(G, B)$ which descends to a pointed map $f_*\colon \mathrm{H}^1(G, A) \to \mathrm{H}^1(G, B)$. Thus we have functors

$$\mathrm{H}^0(G, -)\colon \mathbf{Grp}^{\mathrm{B}G} \to \mathbf{Grp} \quad \text{and} \quad \mathrm{H}^1(G, -)\colon \mathbf{Grp}^{\mathrm{B}G} \to \mathbf{Set}_*,$$

where $\mathbf{Grp}^{\mathrm{B}G}$ is the category of G-groups together with G-equivariant group homomorphism as morphisms, written here as a functor category.

3.2.4. Proposition. — *Let G be a group and let $1 \to A \xrightarrow{i} B \xrightarrow{p} C \to 1$ be a short exact sequence of G-groups. Then there exists a* connecting map $\delta\colon C^G \to \mathrm{H}^1(G, A)$ *such that*

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(G, A) &\xrightarrow{i_*} \mathrm{H}^0(G, B) \xrightarrow{p_*} \mathrm{H}^0(G, C) \\ \xrightarrow{\delta} \mathrm{H}^1(G, A) &\xrightarrow{i_*} \mathrm{H}^1(G, B) \xrightarrow{p_*} \mathrm{H}^1(G, C) \end{aligned}$$

becomes an exact sequence of pointed sets (i.e. kernels coincide with images) which is functorial with respect to morphism of short exact sequences of G-groups.

Proof. — See [BerIGH, §§II.4.1–II.4.2] for details. We only explain the definition of the connecting map
$$\delta\colon C^G \to \mathrm{H}^1(G, A).$$
Let $c \in C^G$ and pick some preimage $b \in p^{-1}(\{c\})$. Then $p(b^{-1}(g.b)) = 1$ since c is G-invariant, so the element $b^{-1}(g.b)$ lies in the image of i_*. The map
$$\delta(b) := g \mapsto i_*^{-1}(b^{-1} \cdot (g.b))$$
is a crossed homomorphism and its class $\delta(c) := [\delta(b)] \in \mathrm{H}^1(G, A)$ is independent of the choice of the preimage b. Note that δ is a pointed map since we can pick for $c = 1$ the preimage $b = 1$. □

3.2.5. Remark. — Comparing the constructions from ¶3.2.2 with those of ¶3.1.9, we see that non-abelian group cohomology recovers abelian group cohomology whenever A is an abelian group.

3.2.6. — In the case that A is an abelian group, we know that $\mathrm{H}^1(G, A)$ is also an abelian group, but the connecting map δ does not need to be a group homomorphism.

We will explain in Proposition 3.2.7 that instead δ will be a crossed homomorphism for the following action: The group C acts from the right on A via inner automorphisms of B. That is, for $c \in C$ and $a \in A$, choose $b \in p^{-1}(c)$ and define
$$a.c := b^{-1}ab := i^{-1}(b^{-1}i(a)b).$$
Using that A is abelian, one checks that this is independent of the choice of preimage b. Since for $g \in G$ one has $g.(a.c) = (g.a).(g.c)$, this right action is G-equivariant as long as $c \in \mathrm{H}^0(G, C) = C^G$. So, by functoriality, it induces a right action of $\mathrm{H}^0(G, C)$ on $\mathrm{H}^1(G, A)$. Concretely, $c \in \mathrm{H}^0(G, C)$ acts on a class $[\varphi]$ given by a crossed homomorphism φ via
$$[\varphi].c = [g \mapsto \varphi(g).c].$$

3.2.7. Proposition. — *Let G be a group and let $0 \to A \to B \to C \to 1$ be a short exact sequence of G-groups where we assume that A is an abelian group. Then the connecting map $\delta\colon \mathrm{H}^0(G, C) \to \mathrm{H}^1(G, A)$ is a crossed homomorphism, i.e.*
$$\delta(cc') = \delta(c).c' + \delta(c')$$
for $c, c' \in \mathrm{H}^0(G, C)$.

Proof. — See [GirCNA, Prop. III.3.4.1] for example. Recall that the map δ sends an element $c \in \mathrm{H}^0(G, C)$ to the class represented by the crossed homomorphism
$$\varphi\colon g \mapsto i^{-1}(b^{-1} \cdot (g.b)),$$
for some choice of $b \in p^{-1}(c)$.

Now pick some $b' \in p^{-1}(c')$ and set $\varphi' := \delta(b')$, then for any $g \in G$

$$\begin{aligned}(bb')^{-1} \cdot (g.(bb')) &= (b')^{-1}b^{-1}(g.b)(g.b') \\ &= (b')^{-1}i(\varphi(g))(g.b') \\ &= i(\varphi(g).c')(b')^{-1}(g.b') \\ &= i(\varphi(g).c') \cdot i(\varphi'(g))\end{aligned}$$

Writing A additively, this means $\delta(cc') = [\varphi].c' + [\varphi']$, as desired. □

3.2.8. Remark. — In particular, when $0 \to A \to B \to C \to 1$ is a central extension, i.e. A is contained in the center of B, then the connecting map δ is a group homomorphism. Furthermore, the 6-term sequence in Proposition 3.2.4 extends to a 7th term through a pointed map $\mathrm{H}^1(G, C) \to \mathrm{H}^2(G, A)$, cf. [BerIGH, §II.4.3].

Next we will consider torsors which are equipped with an additional action by a group G and describe their connection with non-abelian group cohomology sets. See for reference [NSW, §I.2], [BS64, §1] and [SerGC, §I.5]. Let A still be a G-group.

3.2.9. Definition. — A *G-equivariant A-pseudo-torsor* is a set T equipped with a left action by G and a free transitive right action by A, which are compatible in the sense that

$$g.(t.a) = (g.t).(g.a)$$

for every $g \in G$, $a \in A$, $t \in T$. Provided that T is non-empty, we call it a *G-equivariant A-torsor*.

3.2.10. Definition. — A morphism of G-equivariant A-pseudo-torsors is a map of sets which is G- and A-equivariant. More generally, when $f\colon A \to A'$ is a homomorphism of G-groups, then a map $\widetilde{f}\colon T \to T'$ from a G-equivariant A-pseudo-torsor to a G-equivariant A'-pseudo-torsor is called *equivariant* (with respect to f) provided that it is G-equivariant and

$$\widetilde{f}(t.a) = \widetilde{f}(t).f(a)$$

for every $t \in T$ and $a \in A$.

3.2.11. Notation. — Whenever G is the trivial group, we drop "G-equivariant" from the terminology.

3.2.12. Remark. — Consider the category $\mathrm{B}G$ with a single object pt and $\mathrm{Mor}_{\mathrm{B}G}(\mathrm{pt}, \mathrm{pt}) := G$. Then the topos of presheaves $\mathbf{PSh}(\mathrm{B}G)$ on $\mathrm{B}G$ is just the category $\mathbf{Set}^{\mathrm{B}G}$ of sets equipped with a G-action. Now a G-group A is nothing else than a group object in $\mathbf{PSh}(\mathrm{B}G)$, and a G-equivariant A-pseudo-torsor is a A-pseudo-torsor object in $\mathbf{PSh}(\mathrm{B}G)$, i.e. an object $T \in \mathbf{PSh}(\mathrm{B}G)$ together with an action morphism $\mathrm{act}\colon T \times A \to T$ satisfying the action condition that the squares

$$\begin{tikzcd} T \times T \times A \arrow[r, "(\mathrm{act},\mathrm{id})"] \arrow[d, "(\mathrm{id},\mathrm{mult})"'] & T \times A \arrow[d, "\mathrm{act}"] \\ T \times A \arrow[r, "\mathrm{act}"] & T, \end{tikzcd} \qquad \begin{tikzcd} T \times A \arrow[r, "\mathrm{act}"] & T \\ T \arrow[u, "(\mathrm{id},e)"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

commute, and the pseudo-torsor condition that

$$(\mathrm{pr}_1, \mathrm{act})\colon T \times A \xrightarrow{\sim} T \times T$$

is an isomorphism.

3.2.13. Example. — Let $1 \to A \to B \xrightarrow{p} C \to 1$ be a short exact sequence of G-groups. Then the fibers $p^{-1}(\{c\})$ for $c \in C^G$ are typical examples of G-equivariant A-torsors. Indeed, the left G-action on B restricts to the fiber since p is equivariant and c is fixed by G. The right A-action is given by multiplication inside B, and both these actions are compatible since G acts via group homomorphisms.

3.2.14. Proposition. — *Let T be a G-equivariant A-pseudo-torsor, then the fixed-point set T^G is an A^G-pseudo-torsor.*

Proof. — The right action of A^G on T^G is inherited from the action of A on T. It is well-defined, since for $t \in T^G$ and $a \in A^G$, we have

$$g.(t.a) = (g.t).(g.a) = t.a$$

for every $g \in G$. The action is still free and transitive, since for every $t, t' \in T^G$ there exists a unique $a \in A$ such that $t' = t.a$, and we can check that $a \in A^G$. Indeed, for every $g \in G$ we have

$$t.a = t' = g.t' = g.(t.a) = (g.t).(g.a) = t.(g.a),$$

and thus $g.a = a$ by uniqueness of a. □

3.2.15. Proposition. — *Let G be a group and A be a G-group.*

(i) *We have a canonical bijection between $\mathrm{H}^1(G, A)$ and the set of isomorphism classes of G-equivariant A-torsors:*

$$\begin{aligned} \mathrm{H}^1(G, A) &\leftrightarrow \{G\text{-equivariant } A\text{-torsors}\}/\simeq \\ [T] &\leftarrow\!\shortmid T \end{aligned}$$

(ii) *A G-equivariant A-torsor T has a G-fixed point if and only if $[T] = 0$.*

We will spell out and check the details of the constructions giving the bijection, since the proposition plays an important role in the proof of Theorem 6.1.5, and we want to lay bare that the proof is essentially elementary.

Proof. — (i) See [NSW, Prop. 1.2.3] or [BS64, Prop. 1.8] for reference. Let T be a G-equivariant A-torsor, and pick some element $t_0 \in T$. For each $g \in G$ there is a unique $a_g \in A$ such that

$$g.t_0 = t_0.a_g. \tag{3.2.1}$$

Then we calculate

$$t_0.a_{gh} = (gh).t_0 = g.(t_0.a_h) = (g.t_0).(g.a_h) = (t_0.a_g).(g.a_h) = t_0.(a_g.(g.a_h)),$$

which by uniqueness of a_{gh} yields $a_{gh} = a_g.(g.a_h)$. So $g \mapsto a_g$ provides a 1-cocycle which is taken to represent the cohomology class $[T]$. Any other choice of t_0, say t_0', yields an equivalent 1-cocycle. Indeed, there is a unique $a \in A$ such that $t_0' = t_0.a$, so

$$g.t_0' = g.(t_0.a) = (g.t_0).(g.a) = (t_0.a_g).(g.a) = t_0.(a_g(g.a)) = t_0'.a^{-1}.(a_g(g.a))$$

which yields $a_g' = a^{-1}a_g(g.a)$ as desired, showing $[g \mapsto a_g] = [g \mapsto a_g']$.

The other way around, let $[g \mapsto a_g] \in \mathrm{H}^1(G, A)$. Define T as the set A with the right A-action given by multiplication, which is clearly free and transitive, and let $g \in G$ act on $t = a \in T$ as

$$g.t \coloneqq a_g \cdot (g.a). \tag{3.2.2}$$

This is an action since, using $a_1 = 1$, we have $1.t = 1 \cdot (1.a) = a$, and

$$(gh).t = a_{gh} \cdot ((gh).a) = a_g \cdot (g.a_h) \cdot (gh).a = a_g \cdot (g.(a_h \cdot (h.a))) = g.(a_h \cdot (h.a)) = g.(h.t).$$

The required compatibility of the actions is satisfied since for $t = a \in T$

$$g.(t.b) = a_g \cdot (g.(ab)) = a_g \cdot (g.a) \cdot (g.b) = (g.t).(g.b).$$

Any 1-cocycle $g \mapsto a_g'$ which is equivalent to $g \mapsto a_g$, i.e. there exists $a \in A$ such that $a_g' = a^{-1}a_g(g.a)$, yields a torsor T' that is isomorphic to T via the map $t' \mapsto a \cdot t'$. The latter map is clearly A-equivariant and it is G-equivariant since for $t' = b \in T'$

$$g.t' = a_g' \cdot (g.b) = a^{-1} \cdot a_g \cdot (g.a) \cdot (g.b) = a^{-1} \cdot a_g \cdot (g.(ab)) = a^{-1} \cdot g.(a \cdot t').$$

These two constructions described above are inverse to each other: Starting with a cocycle $g \mapsto a_g$ yields a torsor T with underlying set A. Picking $t_0 = 1 \in A$ recovers the cocycle $g \mapsto a_g$, since by construction

$$g.t_0 = a_g \cdot (g.1) = a_g = t_0 \cdot a_g.$$

On the other hand, starting with a torsor T and an element $t_0 \in T$, we get an isomorphism $A \xrightarrow{\sim} T$ of A-torsors, mapping $a \mapsto t_0.a$. This map is G-equivariant for the actions constructed above, since

$$g.(t_0.a) = (g.t_0).(g.a) = (t_0.a_g).(g.a) = t_0.(a_g \cdot (g.a)).$$

(ii) Regarding fixed points: Let $t_0 \in T$. Then by (3.2.1) the cohomology class $[T]$ is represented by the 1-cocycle $g \mapsto a_g$ where $g.t_0 = t_0.a_g$. If $t_0 \in T^G$, then we have $g.t_0 = t_0$, which implies $a_g = 1$ for every $g \in G$, i.e. $g \mapsto a_g$ is the trivial 1-cocycle.

Conversely, assume $[T] = 0$, i.e. $g \mapsto a_g$ is equivalent to the trivial 1-cocycle. This means that there exists an $a \in A$ such that $a_g = a \cdot (g.a)^{-1}$ for every $g \in G$. Finally, the element $t_0' \coloneqq t_0.a$ is a G-fixed point, since

$$g.t_0' = g.(t_0.a) = (g.t_0).(g.a) = (t_0.a_g).(g.a) = t_0.(a \cdot (g.a)^{-1} \cdot (g.a)) = t_0.a = t_0'. \qquad \square$$

3.2.16. Remark. — Under the identification of Proposition 3.2.15 the connecting map $\delta \colon \mathrm{H}^0(G, C) \to \mathrm{H}^1(G, A)$ of Proposition 3.2.4 maps a fixed point $c \in C^G$ to the G-equivariant A-torsor $p^{-1}(\{c\})$ of Example 3.2.13, cf. [NSW, p. 17].

3.2.17. Proposition. — *Let $f \colon A \to A'$ be a homomorphism of G-groups, let T be a G-equivariant A-torsor and let T' be a G-equivariant A'-torsor. Then there exists an equivariant map $\widetilde{f} \colon T \to T'$ as in Definition 3.2.10 if and only if $f_* \colon \mathrm{H}^1(G, A) \to \mathrm{H}^1(G, A')$ maps $[T]$ to $[T']$.*

Proof. — "⇒" Assume that there exists an equivariant map $\widetilde{f}\colon T \to T'$. Pick some element $t_0 \in T$, and define $t'_0 \coloneqq \widetilde{f}(t_0) \in T'$. Then (3.2.1) says that $[T]$ is represented by the cocycle $g \mapsto a_g$ where $g.t_0 = t_0.a_g$. Now $[T']$ is represented by the cocycle $g \mapsto f(a_g)$, since

$$g.t'_0 = g.\widetilde{f}(t_0) = \widetilde{f}(g.t_0) = \widetilde{f}(t_0.a_g) = \widetilde{f}(t_0).f(a_g) = t'_0.f(a_g).$$

This is exactly a cocycle representing $f_*([g \mapsto a_g])$, as desired.

"⇐" Assume that $f_*[T] = [T']$. Pick some $t_0 \in T$, and represent $[T]$ by the cocycle $g \mapsto a_g$ with $g.t_0 = t_0.a_g$. Since $[T']$ is represented by the cocycle $g \mapsto f(a_g)$, there exists some t'_0 satisfying $g.t'_0 = t'_0.f(a_g)$ for every $g \in G$. Now define the map

$$\widetilde{f}\colon T \to T', \qquad t_0.a \mapsto t'_0.f(a)$$

for $a \in A$. By construction, $\widetilde{f}$ is equivariant relative to $f\colon A \to A'$. To prove G-equivariance, let $g \in G$ and $t \in T$. Then there exists $a \in A$ such that $t = t_0.a$, and we calculate

$$\begin{aligned}\widetilde{f}(g.t) = \widetilde{f}(g.(t_0.a)) = \widetilde{f}((g.t_0).(g.a)) = \widetilde{f}((t_0.a_g).(g.a)) = t'_0.f(a_g).f(g.a)\\ = (g.t'_0).f(g.a) = (g.t'_0).(g.f(a)) = g.(t'_0.f(a)) = g.\widetilde{f}(t_0.a) = g.\widetilde{f}(t),\end{aligned}$$

as desired. □

Part II

Derived equivalences of generalized Kummer varieties

CHAPTER 4

The integral standard representation of S_n

4.1. The integral standard representation of S_n and its dual

In this section we discuss the standard representation of the symmetric group with integral coefficients, as well as its dual representation. This serves as preliminaries for calculations in later sections. We are not aware of a reference which focuses on these two integral representations as we do.

***4.1.1. Definition* (Standard representation).** — Define the abelian group Γ_n as the kernel of the summation map $\Sigma\colon \mathbb{Z}^n \to \mathbb{Z}$,

$$\Gamma_n \coloneqq \ker(\Sigma\colon \mathbb{Z}^n \to \mathbb{Z}).$$

The symmetric group S_n acts on $\mathbb{Z}^n$ by permuting the factors; explicitly we have $\sigma.\mathrm{e}_i \coloneqq \mathrm{e}_{\sigma(i)}$, where e_i denotes the i-th standard basis vector. This can also be written as $\sigma.(a_1, \ldots, a_n) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$. Since the morphism Σ is equivariant when we endow $\mathbb{Z}$ with the trivial action, this induces an action of S_n on Γ_n. The $\mathbb{Z}[\mathrm{S}_n]$-module Γ_n is called the *standard representation* of S_n.

4.1.2. — Note that, there is an isomorphism of abelian groups $\mathbb{Z}^{n-1} \xrightarrow{\simeq} \Gamma_n$ given by

$$(a_1, \ldots, a_{n-1}) \mapsto (a_1, \ldots, a_{n-1}, -a_1 - \cdots - a_{n-1}).$$

So, when A is some abelian group, we get a short exact sequence

$$0 \to A \otimes_{\mathbb{Z}} \Gamma_n \to A \otimes_{\mathbb{Z}} \mathbb{Z}^n \xrightarrow{\Sigma} A \otimes_{\mathbb{Z}} \mathbb{Z} \to 0. \tag{4.1.1}$$

This identifies $A \otimes_{\mathbb{Z}} \Gamma_n$ with the kernel of the morphism $\Sigma\colon A^{\times n} \to A$, when we identify $A \otimes_{\mathbb{Z}} \mathbb{Z}^n$ with $A^{\times n}$. Note that the latter identification is clearly S_n-equivariant, so we get an isomorphism of $\mathbb{Z}[\mathrm{S}_n]$-modules.

***4.1.3. Definition* (Dual standard representation).** — Define the abelian group

$$\Gamma_n^\vee \coloneqq \operatorname{coker}(\Delta\colon \mathbb{Z} \to \mathbb{Z}^n),$$

where Δ is the diagonal map. Then, similar to above, $\Gamma_n^\vee$ becomes a $\mathbb{Z}[\mathrm{S}_n]$-module.

4.1.4. — Analogous to ¶4.1.2, we have an isomorphism of abelian groups $\mathbb{Z}^{n-1} \xrightarrow{\sim} \Gamma^\vee$ given by

$$(a_1, \ldots, a_{n-1}) \mapsto [(a_1, \ldots, a_{n-1}, 0)].$$

So, when A is some abelian group, we get a short exact sequence of $\mathbb{Z}[\mathrm{S}_n]$-modules

$$0 \to A \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\Delta} A \otimes_{\mathbb{Z}} \mathbb{Z}^n \to A \otimes \Gamma_n^\vee \to 0.$$

4.1.5. — The notation $\Gamma_n^\vee$ is justified since $\Delta\colon \mathbb{Z} \to \mathbb{Z}^n$ is the dual homomorphism to $\Sigma\colon \mathbb{Z}^n \to \mathbb{Z}$ under the identifications $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$, $f \mapsto f(1)$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^n$, $f \mapsto (f(\mathrm{e}_i))_i$. So we can identify $\mathrm{coker}(\Delta\colon \mathbb{Z} \to \mathbb{Z}^n)$ with the dual abelian group $\mathrm{Hom}_{\mathbb{Z}}(\Gamma_n, \mathbb{Z})$. That is, we have an isomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^\vee \arrow[r, "\Sigma^\vee"] \arrow[d, "\wr"] & (\mathbb{Z}^n)^\vee \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_{\mathbb{Z}}(\Gamma_n, \mathbb{Z}) \arrow[r] \arrow[d, dotted, "\wr"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\Delta"] & \mathbb{Z}^n \arrow[r] & \Gamma_n^\vee \arrow[r] & 0
\end{tikzcd}$$

Again, this isomorphism is S_n-equivariant. Indeed, when S_n acts on a abelian group A, then S_n acts on $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$ via $\sigma.f = x \mapsto f(\sigma^{-1}.x)$, where we have endowed $\mathbb{Z}$ with the trivial S_n-action. Now the equalities

$$\sigma.\mathrm{e}_i \coloneqq ((\sigma.\mathrm{e}_i^\vee)(\mathrm{e}_j))_j = (\mathrm{e}_i^\vee(\sigma^{-1}.\mathrm{e}_j))_j = (\mathrm{e}_i^\vee(\mathrm{e}_{\sigma^{-1}(j)}))_j = (\delta_{i,\sigma^{-1}(j)})_j = (\delta_{\sigma(i),j})_j = \mathrm{e}_{\sigma(i)}$$

show that the identifications above are equivariant.

4.1.6. Definition. — We call the composition

$$\phi_0\colon \Gamma_n \hookrightarrow \mathbb{Z}^n \twoheadrightarrow \Gamma_n^\vee$$

the *canonical map*.

4.1.7. Proposition. — *The canonical map ϕ_0 induces a short exact sequence of $\mathbb{Z}[\mathrm{S}_n]$-modules*

$$0 \to \Gamma_n \to \Gamma_n^\vee \to \mathbb{Z}/n\mathbb{Z} \to 0, \tag{4.1.2}$$

where the action of S_n on $\mathbb{Z}/n\mathbb{Z}$ is trivial.

Proof. — We have $\Delta(k) \in \Gamma_n$ for some $k \in \mathbb{Z}$ if and only if $nk = 0$, so the canonical map ϕ_0 is injective.

An element $[(k_1, \ldots, k_n)] \in \Gamma_n^\vee$ lies in the image of the map ϕ_0 if and only if $(k_1, \ldots, k_n) + \Delta(a) \in \Gamma_n$ for some $a \in \mathbb{Z}$. This is equivalent to $\sum k_i = -na$, i.e. to n divides $\sum k_i$. So $\phi_0\colon \Gamma_n \hookrightarrow \Gamma_n^\vee$ is an injection of index n; the quotient is generated by $(k, 0, \ldots, 0)$ for $k = 0, \ldots, n-1$, and we see that Seq. (4.1.2) is an exact sequence.

Finally, the action of S_n on $\mathbb{Z}/n\mathbb{Z}$ is trivial: Consider a transposition $\tau \in \mathrm{S}_n$ and $k \in \mathbb{Z}$, then

$$\begin{aligned}
\tau.(k, 0, \ldots, 0) &= (0, \ldots, k, \ldots, 0) \\
&\equiv (0, \ldots, k, \ldots, 0) + (k, 0, \ldots, -k, \ldots, 0) \pmod{\Gamma_n} \\
&= (k, 0, \ldots, 0).
\end{aligned}$$

$\square$

4.1.8. Proposition. — *Let B be an abelian group, endowed with the trivial S_n-action. Then there exists an exact sequence of $\mathbb{Z}[\mathrm{S}_n]$-modules*

$$0 \to B[n] \xrightarrow{\Delta} B \otimes_{\mathbb{Z}} \Gamma_n \xrightarrow{\mathrm{id} \otimes \phi_0} B \otimes_{\mathbb{Z}} \Gamma_n^{\vee} \xrightarrow{\Sigma} B/nB \to 0,$$

where Σ is the summation map modulo n and Δ is the diagonal map corestricted to $B \otimes_{\mathbb{Z}} \Gamma_n \subset B \otimes_{\mathbb{Z}} \mathbb{Z}^n$. In particular, if A is an n-divisible abelian group, then we have exact sequences of $\mathbb{Z}[\mathrm{S}_n]$-modules

$$0 \to A[n] \xrightarrow{\Delta} A \otimes_{\mathbb{Z}} \Gamma_n \xrightarrow{\mathrm{id} \otimes \phi_0} A \otimes_{\mathbb{Z}} \Gamma_n^{\vee} \to 0$$

and

$$0 \to A[n] \xrightarrow{\Delta} A[n] \otimes_{\mathbb{Z}} \Gamma_n \xrightarrow{\mathrm{id} \otimes \phi_0} A[n] \otimes_{\mathbb{Z}} \Gamma_n^{\vee} \xrightarrow{\Sigma} A[n] \to 0.$$

Proof. — The sequence is obtained by tensoring Seq. (4.1.2) with B. The left most term is $B[n]$ because of the description of the kernel of $\mathrm{id} \otimes \phi_0$ at the beginning of the proof of Proposition 4.1.7. The rightmost map satisfies $\Sigma([(a, 0, \ldots, 0)]) = a$ and $\Sigma([(a_i)_i]) = 0$ when $\sum_i a_i = 0$, so we recognize the map Σ as the summation map.

Since A is n-divisible, we have $A/nA = 0$, and we always have $A[n]/nA[n] = A[n]$. So the last two sequences in the statement follow by taking $B = A$ and $B = A[n]$, respectively. □

4.1.9. Proposition. — *For $n \geq 2$ we have*

(i) $\mathrm{Hom}_{\mathbb{Z}[\mathrm{S}_n]}(\Gamma_n, \Gamma_n) = \mathbb{Z} \cdot \mathrm{id}$, *and* $\mathrm{Hom}_{\mathbb{Z}[\mathrm{S}_n]}(\Gamma_n^{\vee}, \Gamma_n^{\vee}) = \mathbb{Z} \cdot \mathrm{id}$.

Assume that $n \geq 3$, then we have

(ii) $\mathrm{Hom}_{\mathbb{Z}[\mathrm{S}_n]}(\Gamma_n, \Gamma_n^{\vee}) = \mathbb{Z} \cdot \phi_0$,

and in particular, Γ_n and $\Gamma_n^{\vee}$ are not isomorphic as $\mathbb{Z}[\mathrm{S}_n]$-modules.

Proof. — (i) Extending scalars to $\overline{\mathbb{Q}}$, both $\Gamma_n \otimes \overline{\mathbb{Q}}$ and $\Gamma_n^{\vee} \otimes \overline{\mathbb{Q}}$ become the usual standard representation of S_n, which is irreducible. So by Schur's lemma, every S_n-equivariant morphism $\phi \colon \Gamma_n \to \Gamma_n$ (respectively $\phi \colon \Gamma_n^{\vee} \to \Gamma_n^{\vee}$) becomes of the form $\phi_{\overline{\mathbb{Q}}} = \lambda \cdot \mathrm{id}$ for some $\lambda \in \overline{\mathbb{Q}}$. But since ϕ is defined integrally, we must have $\lambda \in \mathbb{Z}$ actually.

(ii) Using Seq. (4.1.2), the canonical map ϕ_0 becomes an isomorphism after extending scalars to $\overline{\mathbb{Q}}$. So, as above, every S_n-equivariant morphism $\phi \colon \Gamma_n \to \Gamma_n^{\vee}$ becomes of the form

$$\phi_{\overline{\mathbb{Q}}} = \lambda \cdot \phi_{0,\overline{\mathbb{Q}}}$$

for some $\lambda \in \overline{\mathbb{Q}}$. By looking at the short exact sequence Seq. (4.1.2) and the elementary divisors normal form of ϕ_0, we see that ϕ_0 corresponds to $\mathrm{diag}(1, \ldots, 1, n)$ in suitable bases of Γ_n and $\Gamma_n^{\vee}$. For $n \geq 3$ this forces $\lambda \in \mathbb{Z}$, as desired.

Finally, Γ_n and $\Gamma_n^{\vee}$ cannot be isomorphic as $\mathbb{Z}[\mathrm{S}_n]$-modules, since ϕ_0 is not surjective. □

4.1.10. Remark. — For $n = 2$, the map $\phi_0 \in \mathrm{Hom}(\Gamma_2, \Gamma_2^{\vee})$ is not a generator. In fact both Γ_2 and $\Gamma_2^{\vee}$ are isomorphic to the sign representation. Under these identifications the map ϕ_0 becomes $2 \cdot \mathrm{id}$.

***4.1.11. Proposition* (The dual isogeny $\widehat{\phi}_0$).** — *There exists a unique map $\widehat{\phi}_0\colon \Gamma_n^\vee \to \Gamma_n$ of $\mathbb{Z}[\mathrm{S}_n]$-modules such that $\widehat{\phi}_0 \circ \phi_0 = n\cdot \mathrm{id}$ and $\phi_0 \circ \widehat{\phi}_0 = n \cdot \mathrm{id}$, which we call the* dual isogeny *of ϕ_0.*

Proof. — We construct a 'dual isogeny' $\widehat{\phi}_0\colon \Gamma_n^\vee \to \Gamma_n$ to the canonical map $\phi_0\colon \Gamma_n \to \Gamma_n^\vee$. Consider the diagram

$$\begin{tikzcd} \Gamma_n^\vee \arrow[r, "\phi_0^{-1}"] \arrow[dr, dotted] & \Gamma_n \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, "\cdot n"] & \Gamma_n \otimes_{\mathbb{Z}} \mathbb{Q} \\ & \frac{1}{n}\Gamma_n \arrow[u, hook] \arrow[r, "\cdot n"] & \Gamma_n . \arrow[u, hook] \end{tikzcd}$$

We claim that $\mathrm{im}(\phi_0^{-1}) \subset \frac{1}{n}\Gamma_n$. Indeed, note that we have $n\Gamma_n^\vee \subset \mathrm{im}(\phi_0) \subset \Gamma_n^\vee$ by Sequence (4.1.2). So for every $y \in \Gamma_n^\vee$, there exists $x \in \Gamma_n$ such that $n\cdot y = \phi_0(x)$, i.e. $y = \phi_0(\frac{1}{n}x)$. Now we define

$$\widehat{\phi}_0 \coloneqq n \cdot \phi_0^{-1} \colon \Gamma_n^\vee \to \Gamma_n,$$

and see that

$$\widehat{\phi}_0 \circ \phi_0 = n \cdot \mathrm{id} \quad \text{and} \quad \phi_0 \circ \widehat{\phi}_0 = n \cdot \mathrm{id}\,.$$

Uniqueness follows from the injectivity of ϕ_0. ☐

4.1.12. Remark. — The map $\widehat{\phi}_0\colon \Gamma_n^\vee \to \Gamma_n$ is explicitly given by

$$[(a_i)_i] \mapsto (n \cdot a_i)_i$$

if $\sum_i a_i = 0$, and maps

$$[(1,0,\dots,0)] \mapsto (n-1,-1,\dots,-1).$$

These two mapping rules are compatible since $(n,0,\dots,0) \equiv (n-1,-1,\dots,-1) \in \Gamma_n^\vee$.

4.1.13. Proposition. — *Let A be an n-divisible abelian group, then we have an exact sequence*

$$0 \to A[n] \xrightarrow{\Delta} A[n] \otimes_{\mathbb{Z}} \Gamma_n \xrightarrow{\phi_0} A \otimes_{\mathbb{Z}} \Gamma_n^\vee \xrightarrow{\widehat{\phi}_0} A \otimes_{\mathbb{Z}} \Gamma_n \to 0$$

of $\mathbb{Z}[\mathrm{S}_n]$-modules, where A carries the trivial action.

Proof. — Denote the inclusion $A[n] \hookrightarrow A$ by $\imath$, and note that the induced map

$$\imath \otimes \mathrm{id}\colon A[n] \otimes_{\mathbb{Z}} \Gamma_n^\vee \to A \otimes_{\mathbb{Z}} \Gamma_n^\vee$$

is still injective, since $\Gamma_n^\vee$ is free as an abelian group. Then the exactness at the two left terms follows from Proposition 4.1.8.

By the surjectivity in the second sequence of Proposition 4.1.8 we know that every element $[(a_i)_i] \in A \otimes_{\mathbb{Z}} \Gamma_n^\vee$ satisfies $\sum_i a_i = 0$. So by the explicit form in Remark 4.1.12, the kernel of $\mathrm{id} \otimes \widehat{\phi}_0$ consists of those $[(a_i)_i]$ such that $na_i = 0$. The image of $\imath \otimes \phi_0$ consists of those $[(a_i)_i]$ such that $na_i = 0$ and $\sum_i a_i = 0$, where the latter condition is vacuous.

The map $\mathrm{id} \otimes \widehat{\phi}_0$ is surjective since $\widehat{\phi}_0 \circ \phi_0 = n \cdot \mathrm{id}$ and A is n-divisible. ☐

4.1.14. Proposition. — *For $n \geq 2$ we have* $\mathrm{Hom}_{\mathbb{Z}[\mathrm{S}_n]}(\Gamma_n^\vee, \Gamma_n) = \mathbb{Z} \cdot \widehat{\phi}_0$.

Proof. — Similar to the proof of Proposition 4.1.9, we see that for every homomorphism $\phi \in \mathrm{Hom}_{\mathbb{Z}[\mathrm{S}_n]}(\Gamma_n^\vee, \Gamma_n)$ there exists some $\lambda \in \overline{\mathbb{Q}}$ such that $\phi_{\overline{\mathbb{Q}}} = \lambda \cdot \widehat{\phi}_{0,\overline{\mathbb{Q}}}$. Recall that in suitable integral bases, the map ϕ_0 corresponds to the matrix $\mathrm{diag}(1, \ldots, 1, n)$. So, $\widehat{\phi}_0$ corresponds to the matrix $\mathrm{diag}(n, \ldots, n, 1)$, which forces $\lambda \in \mathbb{Z}$ as desired. $\square$

4.1.15. — Identifying $\Gamma_n^\vee$ with the dual of Γ_n by ¶4.1.5, the canonical "evaluation" morphism

$$\begin{aligned} \mathrm{ev}\colon \Gamma_n &\to \Gamma_n^{\vee\vee} \\ x &\mapsto (\psi \mapsto \psi(x)) \end{aligned}$$

is an isomorphism since $\Gamma_n \simeq \mathbb{Z}^{n-1}$ is a free abelian group.

4.1.16. Proposition. — *The canonical map ϕ_0 and its dual isogeny $\widehat{\phi}_0$ are both symmetric, i.e. we have the identities*

(i) $\phi_0 = \phi_0^\vee \circ \mathrm{ev}$, *and*
(ii) $\widehat{\phi}_0 = \mathrm{ev}^{-1} \circ (\widehat{\phi}_0)^\vee$.

Proof. — (i) For $x, y \in \Gamma_n$ we have

$$\phi_0^\vee(\mathrm{ev}(x))(y) = \mathrm{ev}(x)(\phi_0(y)) = \phi_0(y)(x).$$

We want to compare this with $\phi_0(x)(y)$. Take $x = \tilde{\mathrm{e}}_i$ and $y = \tilde{\mathrm{e}}_j$, where $\tilde{\mathrm{e}}_i := \mathrm{e}_i - \mathrm{e}_n$ is the i-th standard basis vactor of Γ_n. Then

$$\phi_0(\tilde{\mathrm{e}}_i) = \mathrm{e}_i^\vee - \mathrm{e}_n^\vee,$$

so $\phi_0(\tilde{\mathrm{e}}_i)(\tilde{\mathrm{e}}_j) = 1$ if $i \neq j$ and $\phi_0(\tilde{\mathrm{e}}_i)(\tilde{\mathrm{e}}_j) = 2$ if $i = j$. This is evidently independent of the order of i and j.

(ii) Since Γ_n is torsion-free, we can check the claimed equality after extending scalars to $\mathbb{Q}$. By (i) we know that $\phi_0 = \phi_0^\vee \circ \mathrm{ev}$, so $\phi_0^{-1} = \mathrm{ev}^{-1} \circ (\phi_0^\vee)^{-1} = \mathrm{ev}^{-1} \circ (\phi_0^{-1})^\vee$, and hence $n \cdot \phi_0^{-1} = \mathrm{ev}^{-1} \circ (n \cdot \phi_0^{-1})^\vee$, as desired. $\square$

4.2. Group cohomology of the standard representation

In this section we calculate the group cohomology $\mathrm{H}^\bullet(\mathrm{S}_n, \Gamma_n \otimes_{\mathbb{Z}} A)$ in the stable range with arbitrary coefficients in an abelian group A in terms of the group cohomology $\mathrm{H}^\bullet(\mathrm{S}_n, A)$ of the symmetric group. We believe the cohomology computations in this sections are new and original. We have recalled the general theory of group cohomology in §3.1.

By convention an abstract abelian group A becomes a $\mathbb{Z}[\mathrm{S}_n]$-module by endowing it with the trivial S_n-action.

4.2.1. Proposition. — *Let A be an abelian group, then $\mathrm{H}^0(\mathrm{S}_n, \Gamma_n \otimes_{\mathbb{Z}} A) = A[n]$, for $n \geq 2$.*

Proof. — Recall that $\mathrm{H}^0(\mathrm{S}_n, \Gamma_n \otimes_{\mathbb{Z}} A) = (\Gamma_n \otimes_{\mathbb{Z}} A)^{\mathrm{S}_n}$ is the submodule of S_n-fixed points in $\Gamma_n \otimes_{\mathbb{Z}} A$. Note that $(a_1, \dots, a_n) \in A^{\times n}$ is fixed by S_n if and only if $a_1 = \cdots = a_n =: a$. But $(a, \dots, a) \in \Gamma_n \otimes_{\mathbb{Z}} A$ if and only if $n \cdot a = 0$. □

4.2.2. Proposition. — *Let A be an abelian group, then we have $\mathrm{H}^0(\mathrm{S}_n, \Gamma_n^\vee \otimes_{\mathbb{Z}} A) = 0$ for $n \geq 3$, and $\mathrm{H}^0(\mathrm{S}_2, \Gamma_2^\vee \otimes_{\mathbb{Z}} A) = A[2]$ for $n = 2$.*

Proof. — Recall that $\Gamma_n^\vee \otimes_{\mathbb{Z}} A$ is the cokernel of the diagonal map $\Delta\colon A \to A^{\times n}$. Consider a fixed point $[(a_i)_i] \in (\Gamma_n^\vee \otimes_{\mathbb{Z}} A)^{\mathrm{S}_n}$. Let $\tau \in \mathrm{S}_n$ be some transposition, say $\tau = (1\,2)$ to keep the notation simple. Then being fixed by τ means that we have $(a_1, a_2, \dots, a_n) = (a_2, a_1, \dots, a_n) + \Delta(a)$ for some $a \in A$. For $n \geq 3$ this entails $a_3 = a_3 + a$, which implies $a = 0$. So the representative $(a_i)_i$ itself is already fixed by τ. In conclusion, we see that $(a_i)_i = \Delta(a_1)$, which is zero in $\Gamma_n^\vee \otimes_{\mathbb{Z}} A$.

For $n = 2$ we can consider $(a_1, 0)$ without loss of generality. Being fixed by τ means $(a_1, 0) = (0, a_1) + (a, a)$ for some $a \in A$, or equivalently $2a_1 = 0$ and $a = a_1$. □

4.2.3. Remark. — The preceding proposition also follows from Proposition 4.2.7 below.

Recall the unit $\iota\colon A \to \mathrm{Ind}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}(\mathrm{Res}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}(A))$ of the restriction-coinduction adjunction and the counit $\nu\colon \mathrm{Ind}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}(\mathrm{Res}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}(A)) \to A$ of the induction-restriction adjunction from Definition 3.1.20. Also recall the canonical projection $\pi\colon \mathrm{Ind}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}(A) \to A$ from ¶3.1.16.

4.2.4. Proposition. — *Let A be an abelian group (endowed with the trivial S_{n-1}-action), and let $A^{\times n}$ be the permutation S_n-representation. Then we have an isomorphism of $\mathbb{Z}[\mathrm{S}_n]$-modules*

$$\mathrm{Ind}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}(A) \simeq A^{\times n},$$

such that, under this isomorphism,

(i) *the canonical map π corresponds to the the projection $\mathrm{e}_n^\vee$ onto the n-th coordinate,*
(ii) *the morphism ι corresponds to the diagonal map $\Delta\colon A \to A^{\times n}$.*
(iii) *the morphism ν corresponds to the the sum map $\Sigma\colon A^{\times n} \to A$.*

Proof. — Recall that we have $\mathrm{Ind}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}(A) \simeq \mathrm{Map}(\mathrm{S}_{n-1}\backslash \mathrm{S}_n, A)$ with S_n-action given by $\sigma.\phi = x \mapsto \phi(x \circ \sigma)$, since the action on A is trivial. Now S_n acts transitively on the set $\{1, \dots, n\}$ from the right via $k.\sigma \coloneqq \sigma^{-1}(k)$. Then $\mathrm{S}_{n-1} = \mathrm{Stab}(n)$ is the stabilizer of the element n, so we get an isomorphism of right S_n-sets

$$\mathrm{S}_{n-1}\backslash \mathrm{S}_n \xrightarrow{\sim} \{1, \dots, n\}, \quad \mathrm{S}_{n-1}\sigma \mapsto \sigma^{-1}(n).$$

Identifying $\mathrm{Map}(\{1, \dots, n\}, A)$ with $A^{\times n}$, an n-tuple $(a_1, \dots, a_n) \in A^{\times n}$ gets acted on as $\sigma.(a_i)_i = (a_{i.\sigma})_i = (a_{\sigma^{-1}(i)})_i$, exactly as in the permutation representation.

(i) Let us now compute the maps π, ν, and ι under the identifications above. By definition $\pi(\phi) = \phi(\mathrm{id})$, for $\phi \in \mathrm{Ind}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}(A)$. Now under the identification $\mathrm{S}_{n-1}\backslash \mathrm{S}_n \simeq \{1, \dots, n\}$, the element id corresponds to the element n. Denote the element induced by ϕ by $\overline{\phi} \in \mathrm{Map}(\{1, \dots, n\}, A)$. Then we have $\pi(\overline{\phi}) = \overline{\phi}(n)$, which means that π becomes the projection onto the n-th coordinate.

(ii) Consider the map $\iota\colon A \to \mathrm{Ind}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}(A)$. We have $\iota(a) = (\sigma \mapsto \sigma.a) =: \phi$ by definition. Since the action on A is trivial, this is just the constant map const_a with value a. But this implies that also $\overline{\phi} = \mathrm{const}_a$, which corresponds to the n-tuple $(a, \dots, a) \in A^{\times n}$, as desired.

(iii) Consider the map $\nu\colon \mathrm{Ind}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}(A) \to A$, which is given by

$$\phi \mapsto \sum_{[\sigma] \in \mathrm{S}_n/\mathrm{S}_{n-1}} \sigma.\phi(\sigma^{-1}) = \sum_{[\sigma] \in \mathrm{S}_n/\mathrm{S}_{n-1}} \phi(\sigma^{-1}).$$

If $\{\sigma_1, \dots, \sigma_n\}$ is a set of left-coset representatives, then $\{\sigma_1^{-1}, \dots, \sigma_n^{-1}\}$ is a set of right-coset representatives. So an element $(a_i)_i \in A^{\times n}$ corresponds to ϕ satisfying $\phi(\sigma_i^{-1}) = a_i$. Now $\nu(\phi) = \sum_{i=1}^n \phi(\sigma_i^{-1}) = \sum_{i=1}^n a_i$, which witnesses that ν becomes the summation map Σ. □

Now we can deduce the following result from Nakaoka's stability theorem (Theorem 3.1.32).

4.2.5. Proposition. — *Let A be an abelian group. Then we have a short exact sequence*

$$0 \to \mathrm{H}^{k-1}(\mathrm{S}_n, A)/n\,\mathrm{H}^{k-1}(\mathrm{S}_n, A) \to \mathrm{H}^k(\mathrm{S}_n, \Gamma_n \otimes_{\mathbb{Z}} A) \to \mathrm{H}^k(\mathrm{S}_n, \mathbb{Z}^n \otimes_{\mathbb{Z}} A)[n] \to 0$$

for $k < n/2$. When $k = n/2$, we still have the injection on the left side.

Proof. — Apply group cohomology to the defining short exact sequence (4.1.1) of $\Gamma_n \otimes_{\mathbb{Z}} A$ to get the exact sequence

$$\cdots \to \mathrm{H}^i(\mathrm{S}_n, \Gamma_n \otimes_{\mathbb{Z}} A) \to \mathrm{H}^i(\mathrm{S}_n, \mathbb{Z}^n \otimes_{\mathbb{Z}} A) \xrightarrow{\Sigma_*^i} \mathrm{H}^i(\mathrm{S}_n, \mathbb{Z} \otimes_{\mathbb{Z}} A) \to \cdots$$

Using the identification from Proposition 4.2.4, Shapiro's isomorphism (Proposition 3.1.18) and Proposition 3.1.24, we extend this to a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^i(\mathrm{S}_n, \Gamma_n \otimes_{\mathbb{Z}} A) \arrow[r] & \mathrm{H}^i(\mathrm{S}_n, \mathbb{Z}^n \otimes_{\mathbb{Z}} A) \arrow[r, "\Sigma_*^i"] \arrow[d, "\mathrm{sh}"', "\wr"] & \mathrm{H}^i(\mathrm{S}_n, \mathbb{Z} \otimes_{\mathbb{Z}} A) \arrow[r] & \cdots \\
& & \mathrm{H}^i(\mathrm{S}_{n-1}, \mathbb{Z} \otimes_{\mathbb{Z}} A). \arrow[ur, "\mathrm{cor}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}"'] & &
\end{tikzcd}$$

Now using the formula $\mathrm{cor}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n} \circ \mathrm{res}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n} = (\mathrm{S}_n : \mathrm{S}_{n-1})\,\mathrm{id} = n\,\mathrm{id}$ of Proposition 3.1.25, and that $\mathrm{res}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}$ is an isomorphism for $i < n/2$ by Theorem 3.1.32, we get for $i = k$

$$\ker(\mathrm{cor}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n} : \mathrm{H}^k(\mathrm{S}_{n-1}, A) \to \mathrm{H}^k(\mathrm{S}_n, A)) = \mathrm{H}^k(\mathrm{S}_{n-1}, A)[n],$$

which determines $\ker(\Sigma_*^k) = \mathrm{H}^k(\mathrm{S}_n, \mathbb{Z}^n \otimes_{\mathbb{Z}} A)[n]$. For $i = k-1$ we get

$$\mathrm{im}(\Sigma_*^{k-1}) = \mathrm{im}(\mathrm{cor}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n} : \mathrm{H}^{k-1}(\mathrm{S}_{n-1}, A) \to \mathrm{H}^{k-1}(\mathrm{S}_n, A)) = n \cdot \mathrm{H}^{k-1}(\mathrm{S}_n, A).$$

So the long exact sequence above induces the desired short exact sequence. The statement about injectivity in the case $k = n/2$ is also explained by the part above where we take $i = k-1$. □

4.2.6. Corollary. — *Let A be an abelian group, and assume $n \geq 3$, then we have a short exact sequence*

$$0 \to A/nA \to \mathrm{H}^1(\mathrm{S}_n, \Gamma_n \otimes_{\mathbb{Z}} A) \to A[2][n] \to 0,$$

where $A[2][n]$ denotes the n-torsion subgroup of the 2-torsion subgroup of A.

Proof. — Since the action on A is trivial, we have $\mathrm{H}^0(\mathrm{S}_n, A) = A$. Taking the viewpoint that the first cohomology group consists of crossed-homomorphisms, cf. (3.1.5), we see that

$$\mathrm{H}^1(\mathrm{S}_{n-1}, A) \simeq \mathrm{Hom}((\mathrm{S}_{n-1})^{\mathrm{ab}}, A) \simeq \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, A) \simeq A[2].$$

Now we use Shapiro's isomorphism $\mathrm{H}^1(\mathrm{S}_n, A^{\times n}) \xrightarrow{\sim} \mathrm{H}^1(\mathrm{S}_{n-1}, A)$, and conclude by applying Proposition 4.2.5. □

4.2.7. Proposition. — *Let A be an abelian group. Then we have for $k < n/2 - 1$ the identity*

$$\mathrm{H}^k(\mathrm{S}_n, \Gamma_n^\vee \otimes_{\mathbb{Z}} A) = 0,$$

and for $k < n/2$ we have an exact sequence

$$0 \to \mathrm{H}^k(\mathrm{S}_n, \Gamma_n^\vee \otimes_{\mathbb{Z}} A) \to \mathrm{H}^{k+1}(\mathrm{S}_n, \mathbb{Z} \otimes_{\mathbb{Z}} A) \xrightarrow{\mathrm{res}} \mathrm{H}^{k+1}(\mathrm{S}_{n-1}, \mathbb{Z} \otimes_{\mathbb{Z}} A).$$

Proof. — Similar to above, apply group cohomology to the defining short exact cokernel sequence of $\Gamma_n^\vee \otimes_{\mathbb{Z}} A$, cf. Definition 4.1.3, to get the commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^i(\mathrm{S}_n, \mathbb{Z} \otimes_{\mathbb{Z}} A) \arrow[r, "\Delta_*"] \arrow[dr, "\mathrm{res}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}"'] & \mathrm{H}^i(\mathrm{S}_n, \mathbb{Z}^n \otimes_{\mathbb{Z}} A) \arrow[r] \arrow[d, "\mathrm{sh}", "\wr"'] & \mathrm{H}^i(\mathrm{S}_n, \Gamma_n^\vee \otimes_{\mathbb{Z}} A) \arrow[r] & \cdots \\
 & & \mathrm{H}^i(\mathrm{S}_{n-1}, \mathbb{Z} \otimes_{\mathbb{Z}} A). & &
\end{tikzcd}$$

By Theorem 3.1.32, the restriction map $\mathrm{res}_{\mathrm{S}_{n-1}}^{\mathrm{S}_n}$ is an isomorphism for $i < n/2$. This information for $i = k$ and $i = k+1$ yields the claims in the proposition. □

4.2.8. Proposition. — *Let A be an abelian group. Then we have*

$$\mathrm{H}^1(\mathrm{S}_n, \Gamma_n^\vee \otimes_{\mathbb{Z}} A) = 0$$

when $n = 3$ or $n \geq 5$. For $n = 4$ we have

$$\mathrm{H}^1(\mathrm{S}_4, \Gamma_4^\vee \otimes_{\mathbb{Z}} A) \simeq A[2].$$

Proof. — The claim for $n \geq 5$ follows directly from Proposition 4.2.7, which also provides the following information

$$\begin{aligned}
\mathrm{H}^1(\mathrm{S}_3, \Gamma_3^\vee \otimes_{\mathbb{Z}} A) &= \ker(\mathrm{res}\colon \mathrm{H}^2(\mathrm{S}_3, A) \to \mathrm{H}^2(\mathrm{S}_2, A)),\\
\mathrm{H}^1(\mathrm{S}_4, \Gamma_4^\vee \otimes_{\mathbb{Z}} A) &= \ker(\mathrm{res}\colon \mathrm{H}^2(\mathrm{S}_4, A) \to \mathrm{H}^2(\mathrm{S}_3, A)).
\end{aligned}$$

Since the action on A is trivial, we can apply the universal coefficient theorem for group cohomology (Proposition 3.1.12) to get the short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(\mathrm{S}_n, \mathbb{Z}), A) \xrightarrow{\gamma} \mathrm{H}^2(\mathrm{S}_n, A) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(\mathrm{S}_n, \mathbb{Z}), A) \to 0,$$

which is compatible with the restriction maps $\mathrm{res}^{\mathrm{S}_n}_{\mathrm{S}_{n-1}}$ by functoriality. Recall that $\mathrm{H}_1(\mathrm{S}_n, \mathbb{Z}) \simeq (\mathrm{S}_n)^{\mathrm{ab}}$ by (3.1.6).

For $n \leq 3$ we have a trivial Schur multiplier $\mathrm{H}_2(\mathrm{S}_n, \mathbb{Z}) = 0$, cf. Example 3.1.30, so γ is an isomorphism. The inclusion $\mathrm{S}_2 \hookrightarrow \mathrm{S}_3$ induces an isomorphism $(\mathrm{S}_2)^{\mathrm{ab}} \xrightarrow{\sim} (\mathrm{S}_3)^{\mathrm{ab}}$, and hence an isomorphism of Ext-groups. In conclusion, the restriction map $\mathrm{res}\colon \mathrm{H}^2(\mathrm{S}_3, A) \to \mathrm{H}^2(\mathrm{S}_2, A)$ is an isomorphism, and $\mathrm{H}^1(\mathrm{S}_3, \Gamma_3^\vee \otimes_{\mathbb{Z}} A) = 0$.

Now we consider the case $n = 4$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}((\mathrm{S}_4)^{\mathrm{ab}}, A) \arrow[r] \arrow[d, "\mathrm{res}", "\simeq"'] & \mathrm{H}^2(\mathrm{S}_4, A) \arrow[r] \arrow[d, "\mathrm{res}"] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(\mathrm{S}_4, \mathbb{Z}), A) \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}((\mathrm{S}_3)^{\mathrm{ab}}, A) \arrow[r] & \mathrm{H}^2(\mathrm{S}_3, A) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(\mathrm{S}_3, \mathbb{Z}), A) \arrow[r] & 0.
\end{tikzcd}$$

From this, and the facts $\mathrm{H}_2(\mathrm{S}_3, \mathbb{Z}) = 0$ and $\mathrm{H}_2(\mathrm{S}_4, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, cf. Example 3.1.30, we conclude that

$$\begin{aligned}
\mathrm{H}^1(\mathrm{S}_4, \Gamma_4^\vee \otimes_{\mathbb{Z}} A) &= \ker(\mathrm{res}\colon \mathrm{H}^2(\mathrm{S}_4, A) \to \mathrm{H}^2(\mathrm{S}_3, A))\\
&\simeq \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(\mathrm{S}_4, \mathbb{Z}), A)\\
&\simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, A)\\
&\simeq A[2].
\end{aligned}$$

□

4.2.9. Theorem. — *Let $n \geq 2$, let A be an n-divisible abelian group, and let $\breve{A}$ be an abelian group. We assume that $\breve{A}$ is 2-divisible in the case $n = 2$, . Then we have*

$$\mathrm{H}^1(\mathrm{S}_n, \mathbb{Z}) = 0, \tag{4.2.1}$$

$$\mathrm{H}^1(\mathrm{S}_n, A \otimes_{\mathbb{Z}} \Gamma_n) = \begin{cases} 0 & \text{if } n \text{ odd, or } n = 2\\ A[2] & \text{if } n \text{ even, and } n \neq 2, \end{cases} \tag{4.2.2}$$

$$\mathrm{H}^1(\mathrm{S}_n, \breve{A} \otimes_{\mathbb{Z}} \Gamma_n^\vee) = \begin{cases} 0 & \text{if } n \neq 4\\ \breve{A}[2] & \text{if } n = 4. \end{cases} \tag{4.2.3}$$

Proof. — We have $\mathrm{H}^1(\mathrm{S}_n, \mathbb{Z}) = \mathrm{Hom}(\mathrm{S}_n, \mathbb{Z}) = 0$, since S_n acts trivially on $\mathbb{Z}$ and the latter is torsion free, cf. (3.1.5). For $n \geq 3$, we have by Corollary 4.2.6

$$\mathrm{H}^1(\mathrm{S}_n, A \otimes_{\mathbb{Z}} \Gamma_n) \xrightarrow{\sim} A[2][n],$$

since A is an n-divisible group, i.e. $A/nA = 0$. Of course, $A[2][n]$ is just $A[2]$ when n is even, and 0 when n is odd. We saw in Proposition 4.2.8 that

$$\mathrm{H}^1(\mathrm{S}_n, \check{A} \otimes_{\mathbb{Z}} \Gamma_n^\vee) = 0$$

when $n \geq 5$ or $n = 3$, and

$$\mathrm{H}^1(\mathrm{S}_4, \check{A} \otimes_{\mathbb{Z}} \Gamma_4^\vee) = \check{A}[2]$$

when $n = 4$.

The case $n = 2$ was treated in [Plo05, Prop. 4.8]. Let us briefly spell out the details. Recall that we have an isomorphism $\Gamma_2 \simeq \Gamma_2^\vee$, both being isomorphic to the sign-representation. Now a 1-cocycle $f\colon \mathrm{S}_2 \to A \otimes \Gamma_2$ is determined by the image point $f((1\,2)) = (a, -a)$. But the equality

$$(1\,2).(-a/2, a/2) - (-a/2, a/2) = (a, -a)$$

realized f as a 1-coboundary. □

4.2.10. Remark. — We will apply Theorem 4.2.9 only with $A = \mathcal{A}(\bar{\mathbb{k}})$ the group of $\bar{\mathbb{k}}$-rational points of an abelian variety $\mathcal{A}$ and $\check{A} = \mathcal{A}^\vee(\bar{\mathbb{k}})$ the group of $\bar{\mathbb{k}}$-rational points of the dual abelian variety $\mathcal{A}^\vee$. Recall that $\mathcal{A}(\bar{\mathbb{k}})$ and $\mathcal{A}^\vee(\bar{\mathbb{k}})$ are indeed n-divisible abelian groups for any $n \geq 1$, cf. Example 1.2.13. In this case, using Corollary 5.1.7 below, we can replace the term $\mathcal{A}^\vee(\bar{\mathbb{k}}) \otimes_{\mathbb{Z}} \Gamma_n^\vee$ in the theorem by

$$(\mathcal{A} \otimes \Gamma_n)^\vee(\bar{\mathbb{k}}) \simeq (\mathcal{A}^\vee \otimes \Gamma_n^\vee)(\bar{\mathbb{k}}) \simeq \mathcal{A}^\vee(\bar{\mathbb{k}}) \otimes_{\mathbb{Z}} \Gamma_n^\vee.$$

4.2.11. Remark. — The adventurous reader may ponder whether Theorem 4.2.9 is valid for an abelian variety A without taking rational points and by working in the category of commutative group schemes. They may think in particular about group cohomology with coefficients in this category, for example using explicit resolutions and concrete constructions. Another viewpoint is to consider an abelian variety as an abelian fppf sheaf.

CHAPTER 5

Invariant derived autoequivalences

5.1. S_n-invariant symplectic isomorphisms

From now on let A be an abelian variety. We first discuss the interactions of the construction $A \otimes \Gamma_n$ with taking duals and homomorphism spaces, which is well known to the experts from the viewpoint of Serre's tensor constructions, cf. [Con04, §7], [Ami18, §1]. Afterwards we are concerned with the computation of the S_n-invariants of the group of symplectic automorphisms $\mathrm{Sp}'(A \otimes \Gamma_n)$, cf. Proposition 5.1.10, as well as of the set of symplectic isomorphisms $\mathrm{Sp}'(A \otimes \Gamma_n, A^\vee \otimes \Gamma_n)$, cf. Proposition 5.1.13, which provide a key ingredient in the proofs of our main theorems later on.

5.1.1. Situation. — Let A and B be abelian varieties over a field $\Bbbk$. The reader may assume that $\mathrm{char}(\Bbbk) = 0$ in order to remove the assumptions below that $\lambda_0 \colon A \to A^\vee$ is a *separable* polarization.

5.1.2. Notation. — Denote by $\mathbf{AV}$ the category of abelian varieties, and by $\mathbf{Mod}_{\mathbb{Z}}^{\mathrm{ffg}}$ the category of free and finitely generated abelian groups.

5.1.3. — Let $\mathit{\Gamma}, \mathit{\Gamma}' \in \mathbf{Mod}_{\mathbb{Z}}^{\mathrm{ffg}}$ be free abelian groups of finite rank $\mathrm{rk}(\mathit{\Gamma}) = m$ and $\mathrm{rk}(\mathit{\Gamma}') = m'$. We have non-canonical isomorphisms $\mathit{\Gamma} \simeq \mathbb{Z}^m$ and $\mathit{\Gamma}' \simeq \mathbb{Z}^{m'}$, so

$$A \otimes \mathit{\Gamma} \simeq A^{\times m} \quad \text{and} \quad B \otimes \mathit{\Gamma}' \simeq B^{\times m'}$$

non-canonically. But this viewpoint shows nevertheless that the construction $A \otimes \mathit{\Gamma}$ is functorial with respect to morphisms in $\mathbf{AV}$ and morphisms in $\mathbf{Mod}_{\mathbb{Z}}^{\mathrm{ffg}}$, where we can view the latter homomorphisms as matrices, e.g. $\mathrm{Hom}_{\mathbb{Z}}(\mathit{\Gamma}, \mathit{\Gamma}')$ becomes $\mathrm{Mat}(m' \times m, \mathbb{Z})$ after choosing bases once and for all. In particular, we have a map

$$\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\mathit{\Gamma}, \mathit{\Gamma}') \to \mathrm{Hom}(A \otimes \mathit{\Gamma}, B \otimes \mathit{\Gamma}') \tag{5.1.1}$$

which is natural in each variable. Using the matrix viewpoint and ¶1.2.7, one observes that (5.1.1) is an isomorphism

$$\begin{aligned} \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\mathit{\Gamma}, \mathit{\Gamma}') &\simeq \mathrm{Mat}(m' \times m, \mathrm{Hom}(A, B)) \\ &\simeq \mathrm{Hom}(A \otimes \mathit{\Gamma}, B \otimes \mathit{\Gamma}') \end{aligned}$$

5.1.4. — Recall from ¶1.2.9 that we have a canonical isomorphism

$$\psi\colon (A^{\times m})^\vee \xrightarrow{\sim} (A^\vee)^{\times m}$$

which is defined as $\psi = (\imath_1^\vee, \dots, \imath_m^\vee)$, where $\imath_k\colon A \hookrightarrow A^{\times m}$ denotes the k-th coordinate embedding.

Let Γ be a free abelian group of rank m. We pick a basis of Γ and endow $\Gamma^\vee$ with the dual basis. Then we have an isomorphism

$$\varphi_{A,\Gamma}\colon (A\otimes\Gamma)^\vee \xrightarrow{\sim} (A^{\times m})^\vee \xrightarrow{\sim} (A^\vee)^{\times m} \xrightarrow{\sim} A^\vee \otimes \Gamma^\vee$$

which a priori appears to be non-canonical. But we can demonstrate otherwise:

5.1.5. Proposition. — *The maps $\varphi_{A,\Gamma}\colon (A\otimes\Gamma)^\vee \to A^\vee\otimes\Gamma^\vee$ provide a natural isomorphism of contravariant functors $\mathbf{AV}\times\mathbf{Mod}_{\mathbb{Z}}^{\mathrm{ffg}} \to \mathbf{AV}$.*

Proof. — Let $f\colon A\to B$ be a homomorphism of abelian varieties, and let $g\colon \Gamma\to\Gamma'$ be a homomorphism of free abelian groups of rank m and m', respectively. Identify

$$\Gamma \simeq \mathbb{Z}^m \quad\text{and}\quad \Gamma'\simeq\mathbb{Z}^{m'}$$

using the bases that we have chosen when defining $\varphi_{-,\Gamma}$ and $\varphi_{-,\Gamma'}$. Then g corresponds to a matrix $M\in\mathrm{Mat}(m'\times m,\mathbb{Z})$, and $g^\vee$ corresponds to the transposed matrix M^{t}. Denote by $f\cdot M$ the matrix with entries in $\mathrm{Hom}(A,B)$ that arises from M by multiplying each entry with f. Consider the following diagram:

$$\begin{tikzcd}
(A\otimes\Gamma)^\vee \arrow[r, "\text{basis}"] & (A^{\times m})^\vee \arrow[r, "(\imath_1^\vee,\dots,\imath_m^\vee)"] & (A^\vee)^{\times m} \arrow[r, "\text{dual}", "\text{basis}"'] & A^\vee\otimes\Gamma^\vee \\
(B\otimes\Gamma')^\vee \arrow[u, "(f\otimes g)^\vee"] \arrow[r, "\text{basis}"] \arrow[ur, phantom, "(1)"] & (B^{\times m'})^\vee \arrow[u, "(f\cdot M)^\vee"'] \arrow[r, "(\imath_1^\vee,\dots,\imath_{m'}^\vee)"] \arrow[ur, phantom, "(2)"] & (B^\vee)^{\times m'} \arrow[u, "f^\vee\cdot M^{\mathrm{t}}"'] \arrow[r, "\text{dual}", "\text{basis}"'] \arrow[ur, phantom, "(3)"] & B^\vee\otimes\Gamma'^\vee. \arrow[u, "f^\vee\otimes g^\vee"']
\end{tikzcd}$$

Now (1) and (3) commute be the construction of (5.1.1) in ¶5.1.3. Regarding (2), we post-compose with the projection $\mathrm{pr}_k\colon (A^\vee)^{\times m}\to A^\vee$ onto the k-th factor, denote the k-th standard basis vector of $\mathbb{Z}^m$ by e_k, and calculate

$$\begin{aligned}\mathrm{pr}_k\circ(\imath_1^\vee,\dots,\imath_m^\vee)\circ(f\cdot M)^\vee &= \imath_k^\vee\circ(f\cdot M)^\vee = (f\cdot M\circ\imath_k)^\vee \\ &= (f\cdot M\mathrm{e}_k)^\vee = (M\mathrm{e}_k\circ f)^\vee = f^\vee\circ(M\mathrm{e}_k)^\vee\end{aligned}$$

and

$$\begin{aligned}\mathrm{pr}_k\circ(f^\vee\cdot M^{\mathrm{t}})\circ(\imath_1^\vee,\dots,\imath_{m'}^\vee) &= (f^\vee\cdot \mathrm{e}_k^{\mathrm{t}}M^{\mathrm{t}})\circ(\imath_1^\vee,\dots,\imath_{m'}^\vee) \\ &= f^\vee\circ(M\mathrm{e}_k)^{\mathrm{t}}\circ(\imath_1^\vee,\dots,\imath_{m'}^\vee) = f^\vee\circ(M\mathrm{e}_k)^\vee,\end{aligned}$$

where we used in the very last step that

$$\sum_j M_{j,k}\cdot\imath_j^\vee = \Big(\sum_j M_{j,k}\cdot\imath_j\Big)^\vee. \qquad\square$$

5.1.6. Remark. — Strictly speaking we have defined $A\otimes\Gamma_n$ as the kernel of the summation map $\Sigma\colon A^{\times n}\to A$, and $A\otimes\Gamma_n^\vee$ as the cokernel of the diagonal map $\Delta\colon A\to A^{\times n}$. From this point of view one verifies by calculating that the following diagram commutes, where the top row is exact by ¶1.2.10 and arises by dualizing the sequence (4.1.1) of abelian varieties. (Here f is the inclusion of the kernel of the summation map, and g denotes the canonical map to the cokernel of the diagonal map.)

$$\begin{tikzcd} 0 \arrow[r] & A^\vee \arrow[r, "\Sigma^\vee"] \arrow[d, "\mathrm{id}"] & (A^{\times n})^\vee \arrow[r, "f^\vee"] \arrow[d, "\psi"] & (A \otimes \Gamma_n)^\vee \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & A^\vee \arrow[r, "\Delta"] & (A^\vee)^{\times n} \arrow[r, "g"] & A^\vee \otimes \Gamma_n^\vee \arrow[r] & 0 \end{tikzcd}$$

The isomorphisms ψ and $\varphi = \varphi_{A,\Gamma_n}$ were defined in ¶5.1.4. Our basis of choice for Γ_n is $\mathrm{e}_i - \mathrm{e}_n$, where $i = 1, \dots n-1$, and for Γ_n we have the basis $[\mathrm{e}_i]$, where $i = 1, \dots n-1$. We check that the right square commutes. Post-composition with the projections $\mathrm{pr}_j \colon A^\vee \otimes \Gamma_n^\vee \to A^\vee$ associated with our choice of basis yields for $j = 1, \dots, n-1$ that

$$\begin{aligned} \mathrm{pr}_j \circ g \circ \psi &= (\mathrm{pr}_j - \mathrm{pr}_n) \circ \psi = \iota_j^\vee - \iota_n^\vee, \text{ and} \\ \mathrm{pr}_j \circ \varphi \circ f^\vee &= \iota_j^\vee \circ f^\vee = (f \circ \iota_j)^\vee = (\iota_j - \iota_n)^\vee = \iota_j^\vee - \iota_n^\vee, \end{aligned}$$

as desired. Now we check that the left square commutes. Again, post-composition with the projections $\mathrm{pr}_j \colon (A^\vee)^n \to A^\vee$ for $j = 1, \dots, n$ yields

$$\begin{aligned} \mathrm{pr}_j \circ \psi \circ \Sigma^\vee &= \iota_j^\vee \circ \Sigma^\vee = (\Sigma \circ \iota_j)^\vee = \mathrm{id}^\vee = \mathrm{id}, \text{ and} \\ \mathrm{pr}_j \circ \Delta \circ \mathrm{id} &= \mathrm{id}\,. \end{aligned}$$

One observes readily that ψ is equivariant, since the S_n-action only "permutes the indices". In fact, we already checked in the proof of Proposition 5.1.5 that ψ is natural. This forces φ to be equivariant as well.

5.1.7. Corollary. — *The natural isomorphism* $(A \otimes \Gamma_n)^\vee \xrightarrow{\sim} A^\vee \otimes \Gamma_n^\vee$ *is* S_n-*equivariant.*

Proof. — Recall that the S_n-action on $A \otimes \Gamma_n$ induces a left action on $(A \otimes \Gamma_n)^\vee$ via

$$\sigma.\alpha := ((\sigma^{-1}).)^\vee(\alpha).$$

Under the natural isomorphisms of Proposition 5.1.5 this becomes

$$\mathrm{id}^\vee \otimes ((\sigma^{-1}).)^\vee \colon A^\vee \otimes \Gamma_n^\vee \to A^\vee \otimes \Gamma_n^\vee$$

which matches exactly the S_n-action on $\Gamma_n^\vee$ by ¶4.1.5. □

5.1.8. Corollary. — *We have* S_n-*equivariant natural isomorphisms*

(i) $\mathrm{Hom}(A \otimes \Gamma_n, A \otimes \Gamma_n) \simeq \mathrm{Hom}(A, A) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_n, \Gamma_n)$,
(ii) $\mathrm{Hom}((A \otimes \Gamma_n)^\vee, A \otimes \Gamma_n) \simeq \mathrm{Hom}(A^\vee, A) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_n^\vee, \Gamma_n)$,
(iii) $\mathrm{Hom}(A \otimes \Gamma_n, (A \otimes \Gamma_n)^\vee) \simeq \mathrm{Hom}(A, A^\vee) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_n, \Gamma_n^\vee)$,
(iv) $\mathrm{Hom}((A \otimes \Gamma_n)^\vee, (A \otimes \Gamma_n)^\vee) \simeq \mathrm{Hom}(A^\vee, A^\vee) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_n^\vee, \Gamma_n^\vee)$.

In particular, these isomorphisms are compatible with all well-defined compositions of morphisms in these groups.

Proof. — We spell out the proof for (iii); the other cases are analogous. Apply Corollary 5.1.7 in order to replace $(A \otimes \Gamma_n)^\vee$ by $A^\vee \otimes \Gamma_n^\vee$. We already exhibited a natural isomorphism

$$\mathrm{Hom}(A, A^\vee) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_n, \Gamma_n^\vee) \to \mathrm{Hom}(A \otimes \Gamma_n, A^\vee \otimes \Gamma_n^\vee)$$

in ¶5.1.3, which in particular is compatible with all well-defined compositions of morphisms between A and $A^\vee$ as well as Γ_n and $\Gamma_n^\vee$.

Recall that the induced action on homomorphism spaces is $\sigma.h \coloneqq \sigma. \circ h \circ (\sigma^{-1}).$, so the action on the left hand side is given by

$$\sigma.(f \otimes g) = f \otimes (\sigma. \circ g \circ (\sigma^{-1}).),$$

while the action on the right hand side is given by

$$\begin{aligned} \sigma.(f \otimes g) &= \sigma. \circ (f \otimes g) \circ (\sigma^{-1}). \\ &= (\mathrm{id} \otimes \sigma.) \circ (f \otimes g) \circ (\mathrm{id} \otimes (\sigma^{-1}).) \\ &= f \otimes (\sigma. \circ g \circ (\sigma^{-1}).). \end{aligned}$$

□

5.1.9. Proposition. — *Assume that* $\dim(A) = 2$ *and that* $\mathrm{End}(A) = \mathbb{Z}$. *Write* $\mathrm{Hom}(A, A^\vee) = \mathbb{Z} \cdot \lambda_0$, *where* λ_0 *is a separable polarization, and let* $d^2 = \deg(\lambda_0)$. *Let* Γ *be a free abelian group of rank* $n-1$, *e.g.* $\Gamma = \Gamma_n$, *then we have an isomorphism*

$$\mathrm{Sp}(A \otimes \Gamma) \simeq \left\{ \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \in \mathrm{Sp}(2(n-1), \mathbb{Z}) \;\middle|\; M_3 \equiv 0 \mod d \right\}.$$

Proof. — Let $f \in \mathrm{Sp}(A \otimes \Gamma)$, then by the results of §1.2, using $\dim(A) = 2$ and $\mathrm{End}(A) = \mathbb{Z}$, we have

$$\begin{aligned} f_1 &= \mathrm{id} \otimes g_1 \in \mathrm{Hom}(A, A) \otimes \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \Gamma) \\ f_2 &= \lambda_0^\delta \otimes g_2 \in \mathrm{Hom}(A^\vee, A) \otimes \mathrm{Hom}_{\mathbb{Z}}(\Gamma^\vee, \Gamma) \\ f_3 &= \lambda_0 \otimes g_3 \in \mathrm{Hom}(A, A^\vee) \otimes \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \Gamma^\vee) \\ f_4 &= \mathrm{id} \otimes g_4 \in \mathrm{Hom}(A^\vee, A^\vee) \otimes \mathrm{Hom}_{\mathbb{Z}}(\Gamma^\vee, \Gamma^\vee). \end{aligned}$$

Now the proof is a synthesis of the arguments of Propositions 2.3.5 and 2.3.7. As in these arguments, we can view each g_i as a matrix $M_i \in \mathrm{Mat}((n-1) \times (n-1), \mathbb{Z})$. Since $\lambda_0^\delta \circ \lambda_0 = [d]$ and $\lambda_0 \circ \lambda_0^\delta = [d]$, we get a group homomorphism

$$\begin{aligned} \mathrm{Sp}(A \otimes \Gamma) &\to \mathrm{Mat}(2(n-1) \times 2(n-1), \mathbb{Z}) \\ f &\mapsto \begin{pmatrix} M_1 & M_2 \\ d \cdot M_3 & M_4 \end{pmatrix}. \end{aligned} \tag{5.1.2}$$

As in Proposition 2.3.5 the condition $\widetilde{f} = f^{-1}$ singles out symplectic matrices. So the map in (5.1.2) provides the desired isomorphism. □

Recall from Definition 2.3.6 that the Hecke congruence subgroup $\Gamma_0(l) \subset \mathrm{SL}_2(\mathbb{Z})$ of level $l \in \mathbb{N}$ consists of two-by-two matrices with determinant 1 and lower left entry divisible by l.

5.1.10. Proposition. — *Let $n \geq 3$ be an integer, and let $A \neq 0$ be an abelian variety.*

(i) *Given a symmetric isogeny (e.g. polarization) $\lambda\colon A \to A^\vee$ of exponent e, there exists an associated injective group homomorphism*

$$\Gamma_0(ne) \hookrightarrow \mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}. \tag{5.1.3}$$

(ii) *Assume that $\dim(A) = 2$ and $\mathrm{End}(A) = \mathbb{Z}$, and let d^2 denote the minimal degree of a polarization of A which we assume to be separable. Then (5.1.3) becomes an isomorphism*

$$\mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n} \simeq \Gamma_0(nd) \subset \mathrm{SL}_2(\mathbb{Z}).$$

5.1.11. Remark. — The proposition remains true when replacing $\mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}$ by $\mathrm{Sp}(A \otimes \Gamma_n)^{\mathrm{S}_n}$.

Proof. — (i) Recall from ¶1.2.21 that there exists a symmetric isogeny $\lambda^{\mathrm{D}}\colon A^\vee \to A$ such that $\lambda \circ \lambda^{\mathrm{D}} = [e] = \lambda^{\mathrm{D}} \circ \lambda$. Also recall from Definition 4.1.6 and Proposition 4.1.11 the canonical map $\phi_0\colon \Gamma_n \hookrightarrow \mathbb{Z}^n \twoheadrightarrow \Gamma_n^\vee$ and its dual isogeny $\widehat{\phi}_0\colon \Gamma_n^\vee \to \Gamma_n$. By Propositions 4.1.9 and 4.1.14 and Corollary 5.1.8 we have the inclusions of S_n-fixed point groups

$$\begin{aligned} \mathrm{Hom}(A \otimes \Gamma_n, A \otimes \Gamma_n)^{\mathrm{S}_n} &\supseteq \mathrm{Hom}(A, A) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_n, \Gamma_n)^{\mathrm{S}_n} \\ &\simeq \mathrm{Hom}(A, A) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}[\mathrm{S}_n]}(\Gamma_n, \Gamma_n) \qquad (5.1.4) \\ &\supseteq (\mathbb{Z} \cdot \mathrm{id}) \otimes_{\mathbb{Z}} (\mathbb{Z} \cdot \mathrm{id}), \\ \mathrm{Hom}(A \otimes \Gamma_n, (A \otimes \Gamma_n)^\vee)^{\mathrm{S}_n} &\supseteq (\mathbb{Z} \cdot \lambda) \otimes_{\mathbb{Z}} (\mathbb{Z} \cdot \phi_0), \qquad (5.1.5) \\ \mathrm{Hom}((A \otimes \Gamma_n)^\vee, A \otimes \Gamma_n)^{\mathrm{S}_n} &\supseteq (\mathbb{Z} \cdot \lambda^{\mathrm{D}}) \otimes_{\mathbb{Z}} (\mathbb{Z} \cdot \widehat{\phi}_0), \qquad (5.1.6) \\ \mathrm{Hom}((A \otimes \Gamma_n)^\vee, (A \otimes \Gamma_n)^\vee)^{\mathrm{S}_n} &\supseteq (\mathbb{Z} \cdot \mathrm{id}) \otimes_{\mathbb{Z}} (\mathbb{Z} \cdot \mathrm{id}). \qquad (5.1.7) \end{aligned}$$

Consider the map $\varphi\colon \Gamma_0(ne) \to \mathrm{Isom}_{\mathrm{AV}}((A \otimes \Gamma_n) \times (A \otimes \Gamma_n)^\vee)^{\mathrm{S}_n}$ given by

$$M := \begin{pmatrix} a_1 & a_2 \\ ne\, a_3 & a_4 \end{pmatrix} \mapsto \begin{pmatrix} a_1 \cdot (\mathrm{id} \otimes \mathrm{id}) & a_2 \cdot (\lambda^{\mathrm{D}} \otimes \widehat{\phi}_0) \\ a_3 \cdot (\lambda \otimes \phi_0) & a_4 \cdot (\mathrm{id} \otimes \mathrm{id}) \end{pmatrix} =: f,$$

which is clearly injective. Since $\lambda \circ \lambda^{\mathrm{D}} = [e]$ and $\lambda^{\mathrm{D}} \circ \lambda = [e]$, as well as $\widehat{\phi}_0 \circ \phi_0 = n$ and $\phi_0 \circ \widehat{\phi}_0 = n$, we see that φ is a group homomorphism, cf. Proposition 2.3.7. As before in the proof of Proposition 2.3.5, and using Proposition 4.1.16, we have

$$\widetilde{f} = \begin{pmatrix} a_4 \cdot (\mathrm{id} \otimes \mathrm{id}) & -a_2 \cdot (\lambda^{\mathrm{D}} \otimes \widehat{\phi}_0) \\ -a_3 \cdot (\lambda \otimes \phi_0) & a_1 \cdot (\mathrm{id} \otimes \mathrm{id}) \end{pmatrix}.$$

Now the condition $f^{-1} = \widetilde{f}$ becomes in matrix form $M^{-1} = J^{\mathrm{t}} M^{\mathrm{t}} J$, cf. Proposition 2.3.5, which just means $\det(M) = 1$ in this instance. It is clear that f is admissible, since id is an isogeny. So the injective homomorphism φ factors over $\mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}$ as

$$\varphi\colon \Gamma_0(ne) \xrightarrow{\sim} \left\{ \begin{pmatrix} a_1 \cdot (\mathrm{id} \otimes \mathrm{id}) & a_2 \cdot (\lambda^{\mathrm{D}} \otimes \widehat{\phi}_0) \\ a_3 \cdot (\lambda \otimes \phi_0) & a_4 \cdot (\mathrm{id} \otimes \mathrm{id}) \end{pmatrix} \middle|\, a_1 a_4 - ne a_2 a_3 = 1 \right\} \subset \mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}.$$

(ii) Write $\operatorname{Hom}(A, A^\vee) = \mathbb{Z} \cdot \lambda_0$, so that $d^2 = \deg(\lambda_0)$. Setting $\lambda := \lambda_0$, we have, using $\dim(A) = 2$, $\operatorname{End}(A) = \mathbb{Z}$ and Proposition 1.2.27, that $\lambda^{\mathrm{D}} = \lambda_0^\delta$ and the exponent $\mathrm{e}(\lambda_0)$ satisfies $\mathrm{e}(\lambda_0) = d$. By Propositions 4.1.9 and 4.1.14 and the results of §1.2 we see, for $n \geq 3$, that the inclusions (5.1.4)–(5.1.7) become isomorphisms

$$\begin{aligned}
\operatorname{Hom}(A \otimes \Gamma_n, A \otimes \Gamma_n)^{\mathrm{S}_n} &\simeq (\mathbb{Z} \cdot \mathrm{id}) \otimes_{\mathbb{Z}} (\mathbb{Z} \cdot \mathrm{id}),\\
\operatorname{Hom}(A \otimes \Gamma_n, (A \otimes \Gamma_n)^\vee)^{\mathrm{S}_n} &\simeq (\mathbb{Z} \cdot \lambda_0) \otimes_{\mathbb{Z}} (\mathbb{Z} \cdot \phi_0),\\
\operatorname{Hom}((A \otimes \Gamma_n)^\vee, A \otimes \Gamma_n)^{\mathrm{S}_n} &\simeq (\mathbb{Z} \cdot \lambda_0^\delta) \otimes_{\mathbb{Z}} (\mathbb{Z} \cdot \widehat{\phi}_0),\\
\operatorname{Hom}((A \otimes \Gamma_n)^\vee, (A \otimes \Gamma_n)^\vee)^{\mathrm{S}_n} &\simeq (\mathbb{Z} \cdot \mathrm{id}) \otimes_{\mathbb{Z}} (\mathbb{Z} \cdot \mathrm{id}).
\end{aligned}$$

This shows that

$$\operatorname{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n} = \left\{ f = \begin{pmatrix} a_1 \cdot (\mathrm{id} \otimes \mathrm{id}) & a_2 \cdot (\lambda_0^\delta \otimes \widehat{\phi}_0) \\ a_3 \cdot (\lambda_0 \otimes \phi_0) & a_4 \cdot (\mathrm{id} \otimes \mathrm{id}) \end{pmatrix} \middle| \; a_i \in \mathbb{Z}, \text{ and } f^{-1} = \widetilde{f} \right\},$$

and we see that the injective homomorphism $\varphi\colon \Gamma_0(nd) \hookrightarrow \operatorname{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}$ associated to λ_0 is surjective. □

5.1.12. Remark. — When $n = 2$, we have to replace $\Gamma_0(2d)$ by $\Gamma_0(d) \simeq \operatorname{Sp}(A)$ in Proposition 5.1.10.(ii), since the canonical map $\phi_0 \in \operatorname{Hom}(\Gamma_2, \Gamma_2^\vee)$ from Definition 4.1.6 is not a generator, cf. Remark 4.1.10. More generally, without assuming $\operatorname{End}(A) = \mathbb{Z}$, the action of S_2 on $\operatorname{Sp}(A \otimes \Gamma_2)$ is trivial, since any homomorphism of abelian varieties commutes with negation.

Now we are interested in the case of an abelian variety A and its dual $A^\vee$ and compute the S_n-invariants of $\operatorname{Sp}'(A \otimes \Gamma_n, A^\vee \otimes \Gamma_n)$.

5.1.13. Proposition. — *Let $n \geq 3$ be an integer.*

(i) *If the abelian variety A admits a symmetric isogeny (e.g. polarization) $\lambda\colon A \to A^\vee$ of exponent $\mathrm{e}(\lambda)$ with*
$$\gcd(n, \mathrm{e}(\lambda)) = 1,$$
then
$$\operatorname{Sp}'(A \otimes \Gamma_n, A^\vee \otimes \Gamma_n)^{\mathrm{S}_n} \neq \emptyset$$
is non-empty.

(ii) *Conversely, assume that $\dim(A) = 2$ and $\operatorname{End}(A) = \mathbb{Z}$, and let d^2 denote the minimal degree of a polarization of A which we assume to be separable, then we have $\operatorname{Sp}'(A \otimes \Gamma_n, A^\vee \otimes \Gamma_n)^{\mathrm{S}_n} \neq \emptyset$ if and only if $\gcd(n, d) = 1$.*

(iii) *In case $\operatorname{Sp}'(A \otimes \Gamma_n, A^\vee \otimes \Gamma_n)^{\mathrm{S}_n}$ is non-empty, it is a (right) torsor under the group $\operatorname{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}$.*

Proof. — (ii) As above in the proof of Proposition 5.1.10, using the results of §§1.2 and 4.1, each $f \in \operatorname{Sp}'(A \otimes \Gamma_n, A^\vee \otimes \Gamma_n)^{\mathrm{S}_n}$ is of the form

$$f = \begin{pmatrix} a_1 \cdot (\lambda_0 \otimes \mathrm{id}) & a_2 \cdot (\mathrm{id} \otimes \widehat{\phi}_0) \\ a_3 \cdot (\mathrm{id} \otimes \phi_0) & a_4 \cdot (\lambda_0^\delta \otimes \mathrm{id}) \end{pmatrix},$$

for some $a_i \in \mathbb{Z}$, where λ_0 is the polarization of minimal degree. Here we identified $A^{\vee\vee}$ with A and $\Gamma_n^{\vee\vee}$ with Γ_n even more aggressively than the convention in Definition 2.3.2 warrants, in order to suppress all evaluation isomorphisms from the notation. Otherwise some instances of A would read $A^{\vee\vee}$ and some instances of id would be ev^{-1} etc. Using Propositions 1.2.26 and 4.1.16 and Theorem 1.2.23, we can write a symplectic isomorphism $\widetilde{f} \in \mathrm{Sp}'(A^\vee \otimes \Gamma_n, A \otimes \Gamma_n)$ as

$$\widetilde{f} = \begin{pmatrix} a_4 \cdot (\lambda_0^\delta \otimes \mathrm{id}) & -a_2 \cdot (\mathrm{id} \otimes \widehat{\phi}_0) \\ -a_3 \cdot (\mathrm{id} \otimes \phi_0) & a_1 \cdot (\lambda_0 \otimes \mathrm{id}) \end{pmatrix}.$$

Keeping in mind that

$$\lambda_0^\delta \circ \lambda_0 = [d] \quad \text{and} \quad \lambda_0 \circ \lambda_0^\delta = [d],$$

as well as

$$\widehat{\phi}_0 \circ \phi_0 = n \quad \text{and} \quad \phi_0 \circ \widehat{\phi}_0 = n,$$

the condition $\widetilde{f} = f^{-1}$, i.e. $\widetilde{f} \circ f = \mathrm{id} = f \circ \widetilde{f}$, becomes

$$\det(f) := a_1 a_4 d - a_2 a_3 n = 1. \tag{5.1.8}$$

Note that Equation (5.1.8) has a solution if and only if $\gcd(n, d) = 1$.

(i) Since $\gcd(n, \mathrm{e}(\lambda)) = 1$, there exists a solution to

$$a_1 a_4 \mathrm{e}(\lambda) - a_2 a_3 n = 1$$

with $a_i \in \mathbb{Z}$. Now the admissible symplectic isomorphism

$$f = \begin{pmatrix} a_1 \cdot (\lambda \otimes \mathrm{id}) & a_2 \cdot (\mathrm{id} \otimes \widehat{\phi}_0) \\ a_3 \cdot (\mathrm{id} \otimes \phi_0) & a_4 \cdot (\lambda^{\mathrm{D}} \otimes \mathrm{id}) \end{pmatrix}$$

witnesses that $\mathrm{Sp}'(A \otimes \Gamma_n, A^\vee \otimes \Gamma_n)^{\mathrm{S}_n}$ is non-empty.

(iii) The claim about torsors can be checked directly. In any case, it follows from Proposition 3.2.14 in view of ¶6.1.2 below. □

5.1.14. Remark. — For $n = 2$, the condition in Proposition 5.1.13 should read $\gcd(1, d) = 1$, which is vacuous. So we have $\mathrm{Sp}'(A \otimes \Gamma_2, A^\vee \otimes \Gamma_2)^{\mathrm{S}_2} \neq \emptyset$, and under the assumptions of (ii) it is a (right) torsor under $\Gamma_0(d)$.

5.2. Invariant derived autoequivalences in Orlov's sequence

We consider Orlov's fundamental short exact sequence for derived equivalences of abelian varieties (Theorem 2.3.8) in the equivariant setup of the previous sections and apply group cohomology to it in order to arrive at the proof of Theorem 3. This section is original.

***5.2.1.* Induced action on $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))$.** — Let G be a group which acts on two varieties X and Y. Then G acts from the left on $\mathbf{D}^{\mathrm{b}}(X)$ via

$$g.\mathcal{F} := (g^{-1})^*\mathcal{F}.$$

Accordingly, the diagonal action of G on $\mathbf{D}^{\mathrm{b}}(X \times Y)$ becomes

$$g.\mathcal{P} = (g^{-1}, g^{-1})^*\mathcal{P},$$

where by slight abuse of notation we write g for both both morphisms $g_X.\colon X \to X$ and $g_Y.\colon Y \to Y$. Recall that for automorphisms $g\colon X \to X$ and $h\colon Y \to Y$ we have by Proposition 2.1.12 the identity

$$\mathrm{FM}_{(g,h)^*\mathcal{P}} = h^* \circ \mathrm{FM}_{\mathcal{P}} \circ g_*.$$

So for $\mathrm{FM}_{\mathcal{P}} \in \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))$, we get

$$\begin{aligned} g.\,\mathrm{FM}_{\mathcal{P}} &= \mathrm{FM}_{(g^{-1},g^{-1})^*\mathcal{P}} \\ &= (g^{-1})^* \circ \mathrm{FM}_{\mathcal{P}} \circ g_*^{-1} \\ &= (g^{-1})^* \circ \mathrm{FM}_{\mathcal{P}} \circ g_*^{-1} \\ &= (g^{-1})^* \circ \mathrm{FM}_{\mathcal{P}} \circ g^*, \end{aligned}$$

using equation (2.1.3) $g_*^{-1} \simeq g^*$ for the isomorphism g.

5.2.2. Remark. — Let G act on three varieties X, Y and Z. The conjugation action in ¶5.2.1 is clearly compatible with composition of Fourier–Mukai functors $\mathrm{FM}_{\mathcal{P}} \in \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y))$ and $\mathrm{FM}_{\mathcal{P}'} \in \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(Y), \mathbf{D}^{\mathrm{b}}(Z))$. Also, if X and Y are abelian varieties with a G-action by homomorphisms, then the induced action on

$$\mathrm{Sp}'(X, Y) \subset \mathrm{Isom}(X \times X^\vee, Y \times Y^\vee)$$

is given by conjugation, cf. ¶3.1.2, and thus as well compatible with composition.

5.2.3. Proposition. — *Let A and B be abelian varieties, then the maps in Seq. (2.3.3)*

$$0 \to \mathbb{Z} \times (A \otimes \Gamma_n) \times (A \otimes \Gamma_n)^\vee \xrightarrow{\iota} \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n)) \xrightarrow{\gamma_{A\otimes\Gamma_n}} \mathrm{Sp}'(A \otimes \Gamma_n) \to 0$$

and, more generally, the map

$$\gamma_{A\otimes\Gamma_n, B\otimes\Gamma_n}\colon \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n), \mathbf{D}^{\mathrm{b}}(B \otimes \Gamma_n)) \to \mathrm{Sp}'(A \otimes \Gamma_n, B \otimes \Gamma_n)$$

are S_n-equivariant.

Proof. — We consider the more general situation where a group G acts on abelian varieties X and X' by homomorphisms. The proposition then follows from the specialization $G = \mathrm{S}_n$ and $X = A \otimes \Gamma_n$ and $X' = B \otimes \Gamma_n$.

Let us first treat the map ι, which sends $k \in \mathbb{Z}$ to the shift functor $[k]$, a point $a \in X$ to the pushforward $(\mathrm{t}_a)_*$ along translation by a, and a point $\alpha \in X^\vee$ to the twist $\mathcal{P}_\alpha \otimes -$ by the algebraically trivial line bundle $\mathcal{P}_\alpha$ corresponding to α. The equalities below refer to equality in $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))$, i.e. natural isomorphisms of functors.

Let $g \in G$. On the one hand, G acts trivially on $\mathbb{Z}$. On the other hand, we have

$$\begin{aligned} g.[k] &= (g^{-1})^* \circ [k] \circ g^* \\ &= (g^{-1})^* \circ g^* \circ [k] \\ &= [k]. \end{aligned}$$

First, we claim that $g.(\mathrm{t}_a)_* = (\mathrm{t}_{g(a)})_*$. Indeed, using $(\mathrm{t}_a)_* = \mathrm{t}^*_{(-a)}$, we have

$$\begin{aligned} g.(\mathrm{t}_a)_* &= (g^{-1})^* \circ \mathrm{t}^*_{(-a)} \circ g^* \\ &= (g \circ \mathrm{t}_{(-a)} \circ g^{-1})^* \\ &= (\mathrm{t}_{-g(a)})^* \\ &= (\mathrm{t}_{g(a)})_*. \end{aligned}$$

Second, we claim that $g.(\mathcal{P}_\alpha \otimes -) = \mathcal{P}_{g.\alpha} \otimes -$. Indeed, recalling that $g.\alpha = (g^{-1})^\vee(\alpha)$, we have

$$\begin{aligned} g.(\mathcal{P}_\alpha \otimes -) &= (g^{-1})^* \circ (\mathcal{P}_\alpha \otimes -) \circ g^* \\ &= ((g^{-1})^*\mathcal{P}_\alpha \otimes (g^{-1})^*(-)) \circ g^* \\ &= (g^{-1})^*\mathcal{P}_\alpha \otimes - \\ &= \mathcal{P}_{(g^{-1})^\vee(\alpha)} \otimes - \\ &= \mathcal{P}_{g.\alpha} \otimes -. \end{aligned}$$

Now we treat the map $\gamma_{X,X'} \colon \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(X')) \to \mathrm{Sp}'(X, X')$. By Example 2.3.11.(ii) we have

$$\gamma_X(g^*) = \begin{pmatrix} g^{-1} & 0 \\ 0 & g^\vee \end{pmatrix}.$$

For $\Phi \in \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(X'))$ let us write $\gamma_{X,X'}(\Phi) = \left(\begin{smallmatrix} f_1 & f_2 \\ f_3 & f_4 \end{smallmatrix}\right)$ and calculate

$$\begin{aligned} \gamma_{X,X'}(g.\Phi) &= \gamma_{X,X'}((g^{-1})^* \circ \Phi \circ g^*) \\ &= \gamma_{X'}((g^{-1})^*) \cdot \gamma_{X,X'}(\Phi) \cdot \gamma_X(g^*) \\ &= \begin{pmatrix} g & 0 \\ 0 & (g^{-1})^\vee \end{pmatrix} \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & g^\vee \end{pmatrix} \\ &= \begin{pmatrix} g \circ f_1 \circ g^{-1} & g \circ f_2 \circ g^\vee \\ (g^{-1})^\vee \circ f_3 \circ g^{-1} & (g^{-1})^\vee \circ f_4 \circ g^\vee \end{pmatrix}. \end{aligned}$$

This is exactly how G acts on $\mathrm{Sp}'(X, X')$. Indeed, recall that $g \in G$ acts on a map ϕ between two G-sets by $g.\phi = g.\circ\phi\circ(g^{-1}).$, and the action of G on $X^\vee$ (respectively $X'^\vee$) is given by $g.\alpha = (g^{-1})^\vee(\alpha)$. Considering $\mathrm{Hom}(X, X')$, $\mathrm{Hom}(X^\vee, X')$, $\mathrm{Hom}(X, X'^\vee)$ and $\mathrm{Hom}(X^\vee, X'^\vee)$ produces exactly the formulas above. □

Using non-abelian group cohomology (see §3.2), we are ready to prove Theorem 3.

***5.2.4. Theorem* (Main Theorem 3).** — *Let A be an abelian variety of dimension $\dim(A) = 2$ over an algebraically closed field $\Bbbk$ of characteristic zero. Assume that $\mathrm{End}(A) = \mathbb{Z}$, and let d^2 denote the minimal degree of a polarization of A.*

(i) *For $n \neq 2, 4$, we have an exact sequence of groups*

$$0 \to \mathbb{Z} \times A[n] \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n))^{\mathrm{S}_n} \to \Gamma_0(nd) \xrightarrow{\delta} A[2][n].$$

(ii) *For $n = 4$, we have an exact sequence of pointed sets (δ might not be a homomorphism)*

$$0 \to \mathbb{Z} \times A[4] \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_4))^{\mathrm{S}_4} \to \Gamma_0(4d) \xrightarrow{\delta} A[2] \times A^\vee[2].$$

(iii) *For $n = 2$, we have an exact sequence of groups*

$$0 \to \mathbb{Z} \times A[2] \times A^\vee[2] \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_2))^{\mathrm{S}_2} \to \Gamma_0(d) \simeq \mathrm{Sp}(A) \to 0.$$

By abuse of notation, we have identified, A with its group $A(\Bbbk)$ of $\Bbbk$-rational points.

5.2.5. Remark. — The assumptions "$\dim(A) = 2$" as well as "$\mathrm{End}(A) = \mathbb{Z}$" only serve to be able to identify $\mathrm{H}^0(\mathrm{S}_n, \mathrm{Sp}'(A \otimes \Gamma_n))$ with certain Hecke congruence groups in the sequences above, and for the analysis of the connecting map. The characteristic zero assumption is inherited from Orlov's sequence, cf. Theorem 2.3.8, where it eventually comes from Bondal–Orlov's criterion for Fourier–Mukai equivalences.

Proof. — Apply non-abelian group cohomology to Orlov's short exact sequence

$$0 \to \mathbb{Z} \times (A \otimes \Gamma_n) \times (A \otimes \Gamma_n)^\vee \to \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n)) \to \mathrm{Sp}'(A \otimes \Gamma_n) \to 0 \quad (5.2.1)$$

to get the exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^0(\mathrm{S}_n, \mathbb{Z} \times (A \otimes \Gamma_n) \times (A \otimes \Gamma_n)^\vee) \to \mathrm{H}^0(\mathrm{S}_n, \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n))) \\ &\to \mathrm{H}^0(\mathrm{S}_n, \mathrm{Sp}'(A \otimes \Gamma_n)) \xrightarrow{\delta} \mathrm{H}^1(\mathrm{S}_n, \mathbb{Z} \times (A \otimes \Gamma_n) \times (A \otimes \Gamma_n)^\vee). \end{aligned}$$

For $n = 3$ and $n \geq 5$, we have computed in §§4.2 and 5.1 (Propositions 4.2.1, 4.2.2 and 5.1.10, Theorem 4.2.9, and Remark 4.2.10)

- $\mathrm{H}^0(\mathrm{S}_n, \mathbb{Z}) = \mathbb{Z}$
- $\mathrm{H}^0(\mathrm{S}_n, A \otimes \Gamma_n) \simeq A[n]$
- $\mathrm{H}^0(\mathrm{S}_n, (A \otimes \Gamma_n)^\vee) = 0$
- $\mathrm{H}^0(\mathrm{S}_n, \mathrm{Sp}'(A \otimes \Gamma_n)) \simeq \Gamma_0(nd)$
- $\mathrm{H}^1(\mathrm{S}_n, \mathbb{Z}) = 0$
- $\mathrm{H}^1(\mathrm{S}_n, A \otimes \Gamma_n) \simeq A[2][n]$
- $\mathrm{H}^1(\mathrm{S}_n, (A \otimes \Gamma_n)^\vee) = 0$

For $n = 4$ the only difference is that

$$\mathrm{H}^1(\mathrm{S}_4, (A \otimes \Gamma_4)^\vee) \simeq A^\vee[2].$$

The case $n = 2$ was treated by Ploog in [Plo05, Prop. 4.8]; it also follows from our remarks and computations in §4.2. The differences are that

- $\mathrm{H}^0(\mathrm{S}_2, (A \otimes \Gamma_2)^\vee) \simeq A^\vee[2]$,
- $\mathrm{H}^0(\mathrm{S}_2, \mathrm{Sp}'(A \otimes \Gamma_2)) = \mathrm{Sp}'(A) \simeq \Gamma_0(d)$,
- $\mathrm{H}^1(\mathrm{S}_2, A \otimes \Gamma_2) = 0$.

It is left to show that, for even $n \neq 4$, the connecting map $\delta\colon \Gamma_0(nd) \to A[2]$ is a group homomorphism, which we are going to show using Proposition 3.2.7. Denote the equivalence associated to $(a,\alpha) \in (A \otimes \Gamma_n) \times (A \otimes \Gamma_n)^\vee$ by $\Phi_{(a,\alpha)} := (\mathrm{t}_a)_* \circ (\mathcal{P}_\alpha \otimes -)$. For any $\Phi \in \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n))$ we have by (2.3.6) that

$$\Phi \circ \Phi_{(a,\alpha)} \circ \Phi^{-1} \simeq \Phi_{\gamma(\Phi)(a,\alpha)}.$$

So for $f \in \mathrm{Sp}'(A \otimes \Gamma_n)$ with $\gamma(\Phi) = f$, we have for the action described in ¶3.2.6

$$\Phi_{(a,\alpha)}.f := \Phi^{-1} \circ \Phi_{(a,\alpha)} \circ \Phi \simeq \Phi_{f^{-1}(a,\alpha)}.$$

This means that the right action of $\mathrm{Sp}'(A \otimes \Gamma_n)$ on $(A \otimes \Gamma_n) \times (A \otimes \Gamma_n)^\vee$ is

$$(a,\alpha).f = f^{-1}(a,\alpha).$$

Taking shift autoequivalences into account, the action is $(k,a,\alpha).f = (k, f^{-1}(a,\alpha))$, so the action of $\mathrm{Sp}'(A \otimes \Gamma_n)$ on $\mathbb{Z}$ (and anyways on $\mathrm{H}^1(\mathrm{S}_n, \mathbb{Z}) = 0$) is already trivial. Now assume $f \in \mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}$, and take any class $[(a,\alpha)] \in \mathrm{H}^1(\mathrm{S}_n, (A \otimes \Gamma_n) \times (A \otimes \Gamma_n)^\vee)$, represented by a cocycle $(a,\alpha)\colon \mathrm{S}_n \to (A \otimes \Gamma_n) \times (A \otimes \Gamma_n)^\vee$. When $n \neq 4$, we know that $\mathrm{H}^1(\mathrm{S}_n, (A \otimes \Gamma_n)^\vee) = 0$, so we can take $\alpha = 0$. The action reads now

$$[(a,0)].f = [(f_1 \circ a, f_3 \circ a)] = [(f_1 \circ a, 0)],$$

where we write $f^{-1} = \left(\begin{smallmatrix} f_1 & f_2 \\ f_3 & f_4 \end{smallmatrix}\right)$. Since $f^{-1} \in \mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}$, we have $f_1 = k_1 \cdot \mathrm{id}$ for some $k_1 \in \mathbb{Z}$ by Proposition 5.1.10, and similarly for f_2, f_3, and f_4. We claim that k_1 is odd, since n is even. Indeed, identifying f^{-1} with an element of $\Gamma_0(nd)$, the condition

$$\det(f^{-1}) = k_1 k_4 - nd k_3 k_2 = 1$$

forces k_1 to be odd since n is already even. Finally, using that $\mathrm{H}^1(\mathrm{S}_n, A \otimes \Gamma_n) \simeq A[2]$ is 2-torsion, we see that f acts trivially. In conclusion, δ is a crossed homomorphism for the trivial action, which makes it a group homomorphism. □

5.2.6. — Let us discuss the connecting map $\delta\colon \Gamma_0(4d) \to A[2] \times A^\vee[2]$ in the case where $n = 4$. Assume $f \in \mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}$ and continue with the notation and setup from the proof of Theorem 5.2.4. We have the action

$$\begin{pmatrix} a \\ \alpha \end{pmatrix}.f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}\begin{pmatrix} a \\ \alpha \end{pmatrix} = \begin{pmatrix} f_1 \circ a + f_2 \circ \alpha \\ f_3 \circ a + f_4 \circ \alpha \end{pmatrix}.$$

The following proposition explains that this action is in general non-trivial. In particular, when d is odd, δ will *not* be a homomorphism unless $\mathrm{im}(\delta) \subset A[2] \times \{0\}$.

5.2.7. Proposition. — *Continuing with the notation from ¶5.2.6 and the proof of Theorem 5.2.4, denote by $(f_i)_*$ the induced maps on first group cohomology. We have*

(i) $(f_1)_* = \mathrm{id}$ *as endomorphisms of* $\mathrm{H}^1(\mathrm{S}_n, A \otimes \Gamma_n)$,

(ii) $(f_4)_* = \mathrm{id}$ *as endomorphisms of* $\mathrm{H}^1(\mathrm{S}_n, (A \otimes \Gamma_n)^\vee)$,

(iii) $(f_3)_* = 0$ *as morphisms* $\mathrm{H}^1(\mathrm{S}_n, A \otimes \Gamma_n) \to \mathrm{H}^1(\mathrm{S}_n, (A \otimes \Gamma_n)^\vee)$,

(iv) $f_2 = k_2 \cdot (\lambda_0^\delta \otimes \widehat{\phi}_0)$ *as morphisms* $(A \otimes \Gamma_n)^\vee \to A \otimes \Gamma_n$, *for some* $k_2 \in \mathbb{Z}$, *and*

$$\ker((\lambda_0^\delta \otimes \widehat{\phi}_0)_*) \simeq \ker(\lambda_0^\delta\colon A^\vee[2] \to A[2]).$$

Proof. — (i) and (ii): As before, the condition $\det(f^{-1}) = 1$ yields that we can view f_1 and f_4 as odd integers, so that we get the identities of cohomology classes $[f_1 \circ a] = [a]$ and $[f_4 \circ \alpha] = [\alpha]$.

(iii) We have $f_3 = k_3 \cdot (\lambda_0 \otimes \phi_0)$ for some $k_3 \in \mathbb{Z}$ by Proposition 5.1.10. We have by Proposition 4.1.8 a short exact sequence

$$0 \to A[n] \xrightarrow{\Delta} A \otimes \Gamma_n \xrightarrow{\mathrm{id} \otimes \phi_0} A \otimes \Gamma_n^\vee \to 0$$

Applying cohomology and using our calculations in Corollary 4.2.6, we get

$$\begin{tikzcd}
\mathrm{H}^1(\mathrm{S}_n, A[n]) \arrow[r, "\Delta_*"] \arrow[d, "\wr"] & \mathrm{H}^1(\mathrm{S}_n, A \otimes \Gamma_n) \arrow[r, "(\mathrm{id} \otimes \phi_0)_*"] \arrow[d, "\wr"] & \mathrm{H}^1(\mathrm{S}_n, A \otimes \Gamma_n^\vee) \\
\mathrm{Hom}(\mathrm{S}_n, A[n]) & A[2][n], &
\end{tikzcd}$$

where the first (vertical) isomorphism recognizes a cocycle $g \mapsto a(g)$ as a homomorphism, and the second isomorphism sends a cocycle $g \mapsto (a_i(g))_i$ to $a_n(\tau)$ for some transposition $\tau \in \mathrm{S}_{n-1} \subset \mathrm{S}_n$. Using

$$\mathrm{Hom}(\mathrm{S}_n, A[n]) \simeq \mathrm{Hom}(\mathrm{S}_n/\mathrm{A}_n, A[n]) \simeq A[n][2]$$

we see that Δ_* is surjective. Hence $(\mathrm{id} \otimes \phi_0)_* = 0$, which implies $(\lambda_0 \otimes \phi_0)_* = 0$.

(iv) As before, by Proposition 5.1.10 we can write the map f_2 as $f_2 = k_2 \cdot (\lambda_0^\delta \otimes \widehat{\phi}_0)$ for some $k_2 \in \mathbb{Z}$. We claim that

$$\ker((\lambda_0^\delta \otimes \mathrm{id})_* \colon \mathrm{H}^1(\mathrm{S}_4, A^\vee \otimes \Gamma_4) \to \mathrm{H}^1(\mathrm{S}_4, A \otimes \Gamma_4))$$

is isomorphic to

$$\ker(\lambda_0^\delta \colon A^\vee[2] \to A[2]).$$

For this, note that the homomorphism $(\lambda_0^\delta \otimes \mathrm{id})_*$ becomes λ_0^δ under the identifications $\mathrm{H}^1(\mathrm{S}_4, A^\vee \otimes \Gamma_4) \simeq A^\vee[2]$ and $\mathrm{H}^1(\mathrm{S}_4, A \otimes \Gamma_4) \simeq A[2]$ of Corollary 4.2.6 (the latter identification is, for concreteness, induced from the composition of the inclusion $\Gamma_n \subset \mathbb{Z}^n$ followed by projection onto the last coordinate and finally evaluation at a transposition.)

Finally we claim that

$$(\mathrm{id} \otimes \widehat{\phi}_0)_* \colon \mathrm{H}^1(\mathrm{S}_4, A^\vee \otimes \Gamma_4^\vee) \to \mathrm{H}^1(\mathrm{S}_4, A^\vee \otimes \Gamma_4)$$

is an isomorphism. For notational reasons we use A in place of $A^\vee$ in the argument. By Proposition 4.1.8 we have an exact sequence

$$0 \to A[n] \xrightarrow{\Delta} A[n] \otimes \Gamma_n \xrightarrow{\phi_0} A[n] \otimes \Gamma_n^\vee \xrightarrow{\Sigma} A[n] \to 0,$$

so we get

$$\begin{aligned}
\mathrm{H}^1(\mathrm{S}_n, (A[n] \otimes \Gamma_n)/A[n]) \xrightarrow{(\phi_0)_*} \mathrm{H}^1(\mathrm{S}_n, A[n] \otimes \Gamma_n^\vee) &\xrightarrow{\Sigma_*} \mathrm{H}^1(\mathrm{S}_n, A[n]) \\
[f \colon \mathrm{S}_n \to A[n] \otimes \Gamma_n^\vee] &\mapsto [\Sigma \circ f].
\end{aligned}$$

For $n = 4$, the middle and right groups are isomorphic to $A[2]$, where the latter isomorphism is given by evaluation at a transposition. One can (tediously) check using the Coxeter–Moore presentation of the symmetric group (Example 3.1.8), that for $a \in A[2]$ the assignment

$$\begin{aligned} f((1\,2)) &\coloneqq [(0,0,a,0)] \\ f((2\,3)) &\coloneqq [(a,0,0,0)] \\ f((3\,4)) &\coloneqq [(0,a,0,0)] \end{aligned}$$

extends to a 1-cocycle on S_4, cf. ¶3.1.7 and Example 3.1.8. (See Listings A.1 and A.2 for a computer-based verification.) By construction, these cocycles witness the surjectivity of Σ_*. So Σ_* becomes an isomorphism and $(\phi_0)_* = 0$. By Proposition 4.1.13 we have an exact sequence

$$0 \to A[n] \xrightarrow{\Delta} A[n] \otimes \Gamma_n \xrightarrow{\phi_0} A \otimes \Gamma_n^\vee \xrightarrow{\widehat{\phi}_0} A \otimes \Gamma_n \to 0.$$

It follows that the map $(\phi_0)_*$ in the induced sequence

$$\mathrm{H}^1(\mathrm{S}_n, (A[n] \otimes \Gamma_n)/A[n]) \xrightarrow{(\phi_0)_*} \mathrm{H}^1(\mathrm{S}_n, A \otimes \Gamma_n^\vee) \xrightarrow{(\widehat{\phi}_0)_*} \mathrm{H}^1(\mathrm{S}_n, A \otimes \Gamma_n)$$

is the zero map as well, since it factors as the previous map $(\phi_0)_*$ followed by the map induced by the inclusion $A[n] \subset A$. We conclude that $(\widehat{\phi}_0)_*$ is injective. For $n = 4$ its domain and codomain are isomorphic to $A[2]$, so it is an isomorphism. $\square$

5.2.8. Question. — For $n \neq 4$, we have seen that $\ker(\delta) \subset \Gamma_0(nd)$ is a normal subgroup, which can be written as the intersection of at most 4 subgroups of index 2, since $A[2]$ is abstractly isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. (N.B. For $n = 4$, the index is still bounded by 16, since the cosets are the fibers of δ.) We wonder if $\ker(\delta) \subset \Gamma_0(nd)$ is a congruence subgroup.

CHAPTER 6

Equivariant derived equivalences

6.1. Invariant derived equivalences via equivariant torsors

We use the notion of equivariant torsors explained in §3.2, and specialize the discussion to the torsors that are relevant to our study of Fourier–Mukai equivalences of generalized Kummer varieties. This section is original.

6.1.1. Situation. — In this section, we work over an algebraically closed field $\Bbbk$ of characteristic zero.

6.1.2. — Let A and B be abelian varieties, then $\mathrm{Sp}'(A,B)$ is a pseudo-torsor under the group $\mathrm{Sp}'(A)$, where the right action is afforded by function composition. Indeed, use that

$$\mathrm{Sp}'(A,B) \subset \mathrm{Isom}(A \times A^\vee, B \times B^\vee) \quad \text{and} \quad \mathrm{Sp}'(A) \subset \mathrm{Aut}(A \times A^\vee)$$

are subsets, and that

$$(g \circ f)^\sim = \widetilde{f} \circ \widetilde{g}$$

for $f \in \mathrm{Sp}'(A)$ and $g \in \mathrm{Sp}'(A,B)$, where the operation $(-)^\sim$ is as in Definition 2.3.2. Similarly the set $\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A), \mathbf{D}^{\mathrm{b}}(B))$ of isomorphism classes of derived equivalences is a pseudo-torsor under the group $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A))$, where the right action is given by functor composition.

Recall that the morphism $\gamma_{A,B}\colon \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A), \mathbf{D}^{\mathrm{b}}(B)) \to \mathrm{Sp}'(A,B)$ from Theorem 2.3.8 is an equivariant map of pseudo-torsors with respect to the homomorphism $\gamma_A\colon \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A)) \to \mathrm{Sp}'(A)$, i.e.

$$\gamma_{A,B}(\varPhi' \circ \varPhi) = \gamma_{A,B}(\varPhi') \circ \gamma_A(\varPhi),$$

or diagrammatically

$$\begin{array}{ccc} \gamma_{A,B}\colon \ \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A), \mathbf{D}^{\mathrm{b}}(B)) & \twoheadrightarrow & \mathrm{Sp}'(A,B) \\ \circlearrowleft & & \circlearrowleft \\ \gamma_A\colon \ \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A)) & \twoheadrightarrow & \mathrm{Sp}'(A). \end{array}$$

6.1.3. — We specialize the situation further and consider first the S_n-equivariant $\mathrm{Sp}'(A\otimes\Gamma_n)$-torsor

$$T_1 \coloneqq \mathrm{Sp}'(A\otimes\Gamma_n, A^\vee\otimes\Gamma_n).$$

The actions are indeed compatible, since S_n acts by conjugation $\sigma.f = \sigma.\circ f\circ(\sigma^{-1}).$ on homomorphism sets, and for $\sigma\in\mathrm{S}_n$, $f\in\mathrm{Sp}'(A\otimes\Gamma_n)$, and $g\in T_1$ we have

$$\sigma.(g\circ f) = \sigma.\circ g\circ f\circ(\sigma^{-1}). = \sigma.\circ g\circ(\sigma^{-1}).\circ\sigma.\circ f\circ(\sigma^{-1}). = (\sigma.g)\circ(\sigma.f).$$

Note that T_1 is non-empty since $A^\vee\otimes\Gamma_n \simeq (A^{\times(n-1)})^\vee$ is the dual abelian variety of $A\otimes\Gamma_n\simeq A^{\times(n-1)}$; a concrete witness is the admissible symplectic isomorphism

$$g = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0\end{pmatrix} \in T_1,$$

which corresponds to the Fourier–Mukai equivalence given by the Poincaré bundle by Example 2.3.11.

Second, consider the S_n-equivariant $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_n))$-torsor

$$T_2 \coloneqq \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_n), \mathbf{D}^{\mathrm{b}}(A^\vee\otimes\Gamma_n)).$$

Again, the actions are compatible since S_n acts by conjugation, and T_2 is non-empty by the surjectivity of

$$\gamma_{(A\otimes\Gamma_n),(A^\vee\otimes\Gamma_n)}\colon T_2\to T_1.$$

6.1.4. Proposition. — *Let A be an abelian variety over $\Bbbk$. Then the set of fixed points* $\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_n), \mathbf{D}^{\mathrm{b}}(A^\vee\otimes\Gamma_n))^{\mathrm{S}_n}$ *is a pseudo-torsor under* $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_n))^{\mathrm{S}_n}$.

Proof. — This is a direct application of Proposition 3.2.14 to the S_n-equivariant torsor T_2. □

6.1.5. Theorem. — *Let $n\geq 3$, and let A be an abelian variety over $\Bbbk$ endowed with a symmetric isogeny (e.g. polarization) $\lambda\colon A\to A^\vee$ of exponent e.*

(i) *Assume n is odd. Then* $\gcd(n,e)=1$ *implies*

$$\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_n), \mathbf{D}^{\mathrm{b}}(A^\vee\otimes\Gamma_n))^{\mathrm{S}_n}\neq\emptyset. \tag{6.1.1}$$

(ii) *Conversely, if* $\dim(A)=2$ *and* $\mathrm{End}(A)=\mathbb{Z}$*, and λ is taken to be the polarization of minimal degree, then* (6.1.1) *implies* $\gcd(n,e)=1$.

Proof. — Using Proposition 3.2.15 we associate to the torsor T_1 the cohomology class

$$[T_1]\in\mathrm{H}^1(\mathrm{S}_n,\mathrm{Sp}'(A\otimes\Gamma_n))$$

and to the torsor T_2 the class

$$[T_2]\in\mathrm{H}^1(\mathrm{S}_n,\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_n))).$$

Then the map $\gamma_{A\otimes\Gamma_n}\colon \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_n))\to\mathrm{Sp}'(A\otimes\Gamma_n))$ induces a map of non-abelian cohomology sets

$$(\gamma_{A\otimes\Gamma_n})_*\colon \mathrm{H}^1(\mathrm{S}_n,\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_n)))\to\mathrm{H}^1(\mathrm{S}_n,\mathrm{Sp}'(A\otimes\Gamma_n)),$$

which sends $[T_2]$ to $[T_1]$. Indeed, we know by Proposition 5.2.3 that

$$\gamma_{(A\otimes\Gamma_n),(A^\vee\otimes\Gamma_n)}\colon T_2\to T_1$$

is an S_n-equivariant map which is equivariant relative to $\gamma_{A\otimes\Gamma_n}$, so we can apply Proposition 3.2.17.

(i) We have seen that $\gcd(n,e)=1$ implies $T_1^{\mathrm{S}_n}\neq\emptyset$, the latter being equivalent to the condition $[T_1]=0$, cf. Propositions 3.2.15 and 5.1.13, thus

$$[T_2]\in\ker((\gamma_{A\otimes\Gamma_n})_*).$$

Using the cohomology sequence associated to Sequence (5.2.1)

$$0\to\mathbb{Z}\times(A\otimes\Gamma_n)\times(A\otimes\Gamma_n)^\vee\to\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_n))\to\mathrm{Sp}'(A\otimes\Gamma_n)\to 0,$$

this means that $[T_2]$ is in the image of $\mathrm{H}^1(\mathrm{S}_n,\mathbb{Z}\times(A\otimes\Gamma_n)\times(A\otimes\Gamma_n)^\vee)$.[(1)] But by Theorem 4.2.9 for n odd, the latter is zero. So we have

$$[T_2]=0$$

and thus $T_2^{\mathrm{S}_n}\neq\emptyset$ by Proposition 3.2.15.

(ii) We have seen in Proposition 5.1.13 that in this case the condition $\gcd(n,e)=1$ is equivalent to $T_1^{\mathrm{S}_n}\neq\emptyset$, and to $[T_1]=0$ by Proposition 3.2.15. Finally, $T_2^{\mathrm{S}_n}\neq\emptyset$ implies $[T_2]=0$, so

$$[T_1]=(\gamma_{A\otimes\Gamma_n})_*[T_2]=0. \qquad\square$$

6.1.6. Remark. —

(i) In part (ii) of Theorem 6.1.5, we can take $\lambda\colon A\to A^\vee$ to be the polarization of minimal degree d. Then $\mathrm{e}(\lambda)^2=d$, and the numerical condition can be read as "$\gcd(n,d)=1$".

(ii) The assumptions on the ground field $\Bbbk$ are inherited from Orlov's theorem (Theorem 2.3.8). Also the vanishing of $\mathrm{H}^1(\mathrm{S}_n,A(\Bbbk)\otimes\Gamma_n)$ in the proof above uses that $A(\Bbbk)$ is n-divisible, which is fine when $\Bbbk$ is algebraically closed.

We are going to use Ploog's method, recalled in §2.2, to enhance an invariant derived equivalence to an equivalence of equivariant derived categories.

6.1.7. — Theorem 2.2.13 specialized to the group $G=\mathrm{S}_n$ yields the diagram

$$\begin{tikzcd}[column sep=-2em] & \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_n),\mathbf{D}^{\mathrm{b}}(A^\vee\otimes\Gamma_n))^{\mathrm{hS}_n} \arrow[dl, "\mathrm{for}"'] \arrow[dr, "\inf_{\Delta\mathrm{S}_n}^{\mathrm{S}_n\times\mathrm{S}_n}"] & \\ \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_n),\mathbf{D}^{\mathrm{b}}(A^\vee\otimes\Gamma_n))^{\mathrm{S}_n} & & \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}_{\mathrm{S}_n}(A\otimes\Gamma_n),\mathbf{D}^{\mathrm{b}}_{\mathrm{S}_n}(A^\vee\otimes\Gamma_n)), \end{tikzcd}$$

which has the following properties.

(i) For $n\geq 3$ the inflation map $\inf_{\Delta\mathrm{S}_n}^{\mathrm{S}_n\times\mathrm{S}_n}$ is injective, since the center $\mathrm{Z}(\mathrm{S}_n)=1$ is trivial, cf. ¶3.1.31.

(ii) For $n=3$ the forgetful map for is surjective, since the Schur multiplier $\mathrm{H}^2(\mathrm{S}_3,\Bbbk^\times)=0$ is trivial, cf. Proposition 3.1.29 and Example 3.1.30.

Furthermore, since $\mathrm{Hom}(\mathrm{S}_n,\Bbbk^\times)=\{\mathrm{id},\mathrm{sgn}\}$, an equivariant structure on an invariant Fourier–Mukai kernel of an equivalence is unique up to the sign representation of S_n.

---

(1) As a reminder, in the kernel term of Orlov's short exact sequence and the sequences derived from it, like Seq. (5.2.1), one denotes the groups of $\Bbbk$-rational points $A(\Bbbk)$ and $A^\vee(\Bbbk)$ just by A and $A^\vee$.

Putting everything together, we are ready to prove the main theorem for generalized Kummer fourfolds. In §6.2 we will explain how to treat generalized Kummer varieties of dimension $2m$ with m even.

***6.1.8. Theorem* (Main Theorem 1 Part 1).** — *Let A be an abelian surface over an algebraically closed field $\Bbbk$ of characteristic zero, which admits a symmetric isogeny (e.g. polarization) $\lambda\colon A \to A^\vee$ of exponent e such that $\gcd(3, e) = 1$. Then the derived categories*

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^2(A)) \quad \textit{and} \quad \mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^2(A^\vee))$$

of 4-dimensional generalized Kummer varieties are derived equivalent.

Proof. — We apply the method of Ploog recalled in ¶6.1.7 above. Theorem 6.1.5 instantiated with $n = 3$ provides a S_3-invariant derived equivalence

$$\mathrm{FM}_{\mathcal{E}}\colon\ \mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_3) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(A^\vee \otimes \Gamma_3),$$

i.e. $\mathrm{FM}_{\mathcal{E}} \in \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_3), \mathbf{D}^{\mathrm{b}}(A^\vee \otimes \Gamma_3))^{\mathrm{S}_3}$. Since the Schur multiplier

$$\mathrm{H}^2(\mathrm{S}_3, \Bbbk^\times) = 0$$

is zero, we can enhance the Fourier–Mukai kernel $\mathcal{E}$ to an equivariant object

$$(\mathcal{E}, \phi) \in \mathbf{D}^{\mathrm{b}}_{\Delta\mathrm{S}_3}((A \otimes \Gamma_3) \times (A^\vee \otimes \Gamma_3)),$$

thus providing an element in $\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_3), \mathbf{D}^{\mathrm{b}}(A^\vee \otimes \Gamma_3))^{\mathrm{hS}_3}$. Now the inflation of the equivariant kernel $(\mathcal{E}, \phi)$ along the diagonal inclusion $\Delta\mathrm{S}_3 \subset \mathrm{S}_3 \times \mathrm{S}_3$ provides an element of $\mathrm{Eq}(\mathbf{D}^{\mathrm{b}}_{\mathrm{S}_3}(A \otimes \Gamma_3), \mathbf{D}^{\mathrm{b}}_{\mathrm{S}_3}(A^\vee \otimes \Gamma_3))$, i.e. a kernel for an equivalence

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{S}_3}(A \otimes \Gamma_3) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathrm{S}_3}(A^\vee \otimes \Gamma_3).$$

Finally, as explained in Proposition 1.1.14, we have

$$\mathrm{Kum}^2(A) \simeq \mathrm{Hilb}_{\mathrm{S}_3}(A \otimes \Gamma_3),$$

and by the derived McKay-correspondence we know

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}_{\mathrm{S}_3}(A \otimes \Gamma_3)) \simeq \mathbf{D}^{\mathrm{b}}_{\mathrm{S}_3}(A \otimes \Gamma_3),$$

and similarly for $A^\vee$ instead of A, cf. Proposition 2.2.21. □

6.1.9. Remark. — We want to discuss Stellari's theorem (Theorem 2.3.17) and the case $n = 2$ in this remark. Let A and B be abelian varieties, and assume n is odd or $n = 2$. Consider the equivariant pseudo-torsors $T_1 \coloneqq \mathrm{Sp}'(A \otimes \Gamma_n, B \otimes \Gamma_n)$ and $T_2 \coloneqq \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n), \mathbf{D}^{\mathrm{b}}(B \otimes \Gamma_n))$. If $T_1^{\mathrm{S}_n} \neq \emptyset$, then the cohomology class $[T_1]$ vanishes. Since $\mathrm{H}^1(\mathrm{S}_n, \mathbb{Z} \times (A \otimes \Gamma_n) \times (A \otimes \Gamma_n)^\vee) = 0$ by Theorem 4.2.9, we see as above that $[T_2] = 0$, i.e. $T_2^{\mathrm{S}_n} \neq \emptyset$. So

$$\gamma^{\mathrm{S}_n}_{A\otimes\Gamma_n, B\otimes\Gamma_n}\colon\ \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n), \mathbf{D}^{\mathrm{b}}(B \otimes \Gamma_n))^{\mathrm{S}_n} \to \mathrm{Sp}'(A \otimes \Gamma_n, B \otimes \Gamma_n)^{\mathrm{S}_n}$$

is an equivariant morphism of torsors, relative to the homomorphism

$$\gamma^{\mathrm{S}_n}_{A\otimes\Gamma_n}\colon\ \mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n))^{\mathrm{S}_n} \to \mathrm{Sp}'(A \otimes \Gamma_n)^{\mathrm{S}_n}.$$

The latter map is surjective by Theorem 5.2.4 and Remark 5.2.5, so the former map is surjective as well. If $n = 2$, one notes that $T_1^{\mathrm{S}_2} = T_1$ since $\mathrm{S}_2 \simeq \mathbb{Z}/2\mathbb{Z}$ acts via conjugation by $-\,\mathrm{id} = \gamma([-1]_*)$ on symplectic isomorphisms. Hence

$$\gamma^{\mathrm{S}_2}_{A\otimes\Gamma_2, B\otimes\Gamma_2}\colon\ \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A\otimes\Gamma_2), \mathbf{D}^{\mathrm{b}}(B\otimes\Gamma_2))^{\mathrm{S}_2} \to \mathrm{Sp}'(A\otimes\Gamma_2, B\otimes\Gamma_2)^{\mathrm{S}_2} = \mathrm{Sp}'(A, B)$$

is surjective. Finally, recall that the set of symplectic isomorphisms on the right hand side is non-empty if and only if A and B are derived equivalent. Also recall that, for $n = 2$ or $n = 3$, Ploog's method, cf. §2.2, can be applied without any difficulties due to the vanishing of the Schur multipliers. In conclusion, we have recovered [Ste07, Prop. 3.1] and a proof of Theorem 2.3.17.

6.2. Equivariant semi-homogeneous vector bundles and Orlov's construction

In this section we prove the remaining cases of Theorem 1. This section is original but builds upon Orlov's constructions in [Orl02] and Mukai's theory of semi-homogeneous vector bundle in [Muk78].

***6.2.1. Theorem* (Main Theorem 1 Part 2).** — *Let $n \geq 5$ be an odd integer and let A be an abelian surface over an algebraically closed field of characteristic zero, which admits a symmetric isogeny (e.g. polarization) $\lambda\colon A \to A^\vee$ of exponent e such that $\gcd(e,n) = 1$. Then the two derived categories*

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^{n-1}(A)) \quad \text{and} \quad \mathbf{D}^{\mathrm{b}}(\mathrm{Kum}^{n-1}(A^\vee))$$

of $2(n-1)$-dimensional generalized Kummer varieties are derived equivalent.

6.2.2. Situation. — Let $n \geq 2$ be a natural number. We work over an algebraically closed field $\Bbbk$, and assume that $\mathrm{char}(\Bbbk)$ does not divide n.

We study Orlov's construction (Construction 2.3.12) concerning preimages of the map $\gamma\colon \mathrm{Eq}(A \otimes \Gamma_n, A^\vee \otimes \Gamma_n) \to \mathrm{Sp}(A \otimes \Gamma_n, A^\vee \otimes \Gamma_n)$ from an equivariant perspective, with the goal to construct an equivariant Fourier–Mukai kernel out of an invariant symplectic isomorphism. On the way we will make use of Mukai's theory of semi-homogeneous vector bundles as recalled in §1.3.

For the readers convenience we reproduce here the steps of Construction 2.3.12.

6.2.3. Construction. — Let A and B be abelian varieties. The following steps construct a preimage under Orlov's map $\gamma\colon \mathrm{Eq}(\mathbf{D}^{\mathrm{b}}(A), \mathbf{D}^{\mathrm{b}}(B)) \to \mathrm{Sp}(A, B)$ of a symplectic isomorphism as in step (1).

(1) Consider some symplectic isomorphism

$$f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in \mathrm{Sp}(A, B)$$

and assume that $f_2\colon A^\vee \to B$ is an isogeny.

(2) Denote by f_2^{-1} the inverse isogeny of f_2 with rational coefficients, and subsequently define the map $g \in \mathrm{Hom}(A \times B, A^\vee \times B^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$g := \begin{pmatrix} f_2^{-1} \circ f_1 & -f_2^{-1} \\ -(f_2^{-1})^\vee & f_4 \circ f_2^{-1} \end{pmatrix}.$$

(3) Since f is symplectic, the map g is symmetric, so there exists a (unique) element

$$\mu := [\mathcal{L}] \otimes \frac{1}{\ell} \in \mathrm{NS}(A \times B) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

such that $g = \varphi_{\mathcal{L}}/\ell$, cf. ¶1.2.11.

(4) Take the semi-homogeneous vector bundle

$$\mathcal{F} := [\ell]_*(\mathcal{L}^{\otimes \ell})$$

on $A \times B$ of slope $\mu(\mathcal{F}) = \mu$, cf. Proposition 1.3.7.(i)

(5) Consider a Jordan–Hölder filtration

$$0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_k = \mathcal{F},$$

where each graded piece $\mathcal{E}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ is a *simple* semi-homogeneous vector bundle of slope $\mu(\mathcal{E}_i) = \mu$, cf. Proposition 1.3.7.(ii).

(6) Finally, take any of the vector bundles $\mathcal{E}_i$ as the kernel of a Fourier–Mukai functor

$$\mathrm{FM}_{\mathcal{E}_i} \colon \mathbf{D}^{\mathrm{b}}(A) \to \mathbf{D}^{\mathrm{b}}(B).$$

Now we specialize our situation to the equivariant setting of the previous sections, that is, we let A_0 be an abelian surface and take

$$A = A_0 \otimes \Gamma_n \quad \text{and} \quad B = A_0^\vee \otimes \Gamma_n$$

with their S_n-action. We follow the steps of Orlov's and Mukai's constructions with the plan to endow $\mathcal{L}$ in step (3) with an equivariant structure which we can carry through the remaining steps to an equivariant structure on some $\mathcal{E}_i$ in step (6).

6.2.4. — Ad (1): Let A_0 be an abelian surface over $\Bbbk$, and let $\lambda \colon A_0 \to A_0^\vee$ be a symmetric isogeny of exponent $\mathrm{e}(\lambda)$. We assume that $\gcd(\mathrm{e}(\lambda), n) = 1$, so we can pick integers n_3 and n_4 which solve the equation

$$n_4 \mathrm{e}(\lambda) - n_3 n = 1.$$

Now recall the maps ϕ_0 and $\widehat{\phi}_0$ from §4.1 and consider the element

$$f \coloneqq \begin{pmatrix} \lambda & \widehat{\phi}_0 \\ n_3\phi_0 & n_4\lambda^{\mathrm{D}} \end{pmatrix} \in \mathrm{Sp}(A_0 \otimes \Gamma_n, A_0^\vee \otimes \Gamma_n).$$

Note that indeed $\widehat{\phi}_0 \coloneqq \mathrm{id} \otimes \widehat{\phi}_0 \colon A_0^\vee \otimes \Gamma_n^\vee \to A_0^\vee \otimes \Gamma_n$ is an isogeny.

6.2.5. — Ad (2): Recall that by construction we have

$$(\widehat{\phi}_0)^{-1} = \frac{1}{n}\phi_0 \in \mathrm{Hom}(A_0^\vee \otimes \Gamma_n, A_0^\vee \otimes \Gamma_n^\vee) \otimes_{\mathbb{Z}} \mathbb{Q},$$

and that $\phi_0^\vee = \phi_0$, so we get

$$g = \begin{pmatrix} \widehat{\phi}_0^{-1} \circ \lambda & -\widehat{\phi}_0^{-1} \\ -(\widehat{\phi}_0^{-1})^\vee & n_4\lambda^{\mathrm{D}} \circ (\widehat{\phi}_0)^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\lambda \otimes \phi_0 & -\frac{1}{n}\,\mathrm{id} \otimes \phi_0 \\ -\frac{1}{n}\,\mathrm{id} \otimes \phi_0 & \frac{n_4}{n}\lambda^{\mathrm{D}} \otimes \phi_0 \end{pmatrix} = \begin{pmatrix} \lambda & -\mathrm{id} \\ -\mathrm{id} & n_4\lambda^{\mathrm{D}} \end{pmatrix} \otimes \frac{1}{n}\phi_0.$$

6.2.6. Proposition. — *Let g be defined as in ¶6.2.5, then there exists a line bundle $\mathcal{L} \in \mathrm{Pic}((A_0 \otimes \Gamma_n) \times (A_0^\vee \otimes \Gamma_n))$ which is*

(i) *S_n-equivariant, and satisfies*

(ii) *$\varphi_{\mathcal{L}} = ng$.*

Proof. — The next proposition (Proposition 6.2.7) applied to $X = A_0 \times A_0^\vee$ shows that we can take

$$\mathcal{L} = \mathcal{L}_0^{\boxtimes n}\Big|_{(A_0 \times A_0^\vee) \otimes \Gamma_n},$$

where $\mathcal{L}_0 \in \operatorname{Pic}(A_0 \times A_0^\vee)$ satisfies

$$\varphi_{\mathcal{L}_0} = \begin{pmatrix} \lambda & -\operatorname{id} \\ -\operatorname{id} & n_4\lambda^{\mathrm{D}} \end{pmatrix}.$$

In particular, $\mathcal{L}$ becomes an S_n-equivariant line bundle since the box-product

$$\mathcal{L}_0^{\boxtimes n} = \operatorname{pr}_1^*\mathcal{L}_0 \otimes \cdots \otimes \operatorname{pr}_n^*\mathcal{L}_0 \in \operatorname{Pic}((A_0 \times A_0^\vee)^{\times n})$$

carries a canonical S_n-equivariant structure, cf. Example 2.2.4. □

6.2.7. Proposition. — *Let X be an abelian variety and let $\mathcal{L}_0 \in \operatorname{Pic}(X)$ be a line bundle. Then we have the equality*

$$\varphi_{(\mathcal{L}_0^{\boxtimes n}|X\otimes\Gamma_n)} = \varphi_{\mathcal{L}_0} \otimes \phi_0 \colon X \otimes \Gamma_n \to X^\vee \otimes \Gamma_n^\vee.$$

Proof. — Let $i\colon X \otimes \Gamma_n \hookrightarrow X \otimes \mathbb{Z}^n \simeq X^n$ be the closed immersion induced by the inclusion $\Gamma_n \hookrightarrow \mathbb{Z}^n$. Consider the diagram

$$\begin{tikzcd} X \otimes \Gamma_n \arrow[r, "\varphi_{i^*(\mathcal{L}_0^{\boxtimes n})}"] \arrow[d, hook, "i"] & (X \otimes \Gamma_n)^\vee \arrow[r, "\sim"] & X^\vee \otimes \Gamma_n^\vee \\ X^n \arrow[r, "\varphi_{\mathcal{L}_0^{\boxtimes n}}"] \arrow[rr, bend right, "\varphi_{\mathcal{L}_0}\times\cdots\times\varphi_{\mathcal{L}_0}"'] & (X^n)^\vee \arrow[u, "i^\vee"] \arrow[r, "\sim"] & (X^\vee)^n. \arrow[u, two heads] \end{tikzcd}$$

The left square commutes by Diagram (1.2.2) in ¶1.2.11. For the right square and lower triangle see ¶4.1.5 and Corollary 5.1.7, which explain in particular that $\Sigma^\vee = \Delta$ and $i^\vee$ becomes the quotient projection in the definition of $\Gamma_n^\vee$, cf. Definition 4.1.3.

It is clear that the square

$$\begin{tikzcd} X \otimes \Gamma_n \arrow[r, "\varphi_{\mathcal{L}_0}\otimes\operatorname{id}"] \arrow[d, hook, "i"] & X^\vee \otimes \Gamma_n \arrow[d, hook] \\ X^n \arrow[r, "\varphi_{\mathcal{L}_0}\times\cdots\times\varphi_{\mathcal{L}_0}"'] & (X^\vee)^n \end{tikzcd}$$

commutes. Finally, the composition

$$X^\vee \otimes \Gamma_n \hookrightarrow (X^n)^\vee \twoheadrightarrow X^\vee \otimes \Gamma_n^\vee$$

equals

$$\operatorname{id} \otimes \phi_0 \colon X^\vee \otimes \Gamma_n \to X^\vee \otimes \Gamma_n^\vee$$

by the definition of ϕ_0, cf. Definition 4.1.6. Putting these facts together yields the claimed result. □

6.2.8. — Ad (3): The symmetric map g from ¶6.2.5 corresponds to

$$\mu \coloneqq \frac{1}{n}[\mathcal{L}] \in \operatorname{NS}((A_0 \otimes \Gamma_n) \times (A_0^\vee \otimes \Gamma_n)) \otimes \mathbb{Q}$$

where $\mathcal{L} \in \operatorname{Pic}((A_0 \otimes \Gamma_n) \times (A_0^\vee \otimes \Gamma_n))$ is the S_n-equivariant line bundle from Proposition 6.2.6, which satisfies $\varphi_{\mathcal{L}} = ng$.

6.2.9. — Ad (4): Now we consider the semi-homogeneous vector bundle

$$\mathcal{F} := [n]_* \mathcal{L}^{\otimes n}.$$

It has slope $\mu(\mathcal{F}) = \mu$ by Proposition 1.3.7. Recall that the S_n-equivariant structure on $\mathcal{L}$ induces one on $\mathcal{L}^{\otimes n}$, cf. ¶2.2.5. Also $\mathcal{F}$ inherits an S_n-equivariant structure from $\mathcal{L}^{\otimes n}$ as the push-forward along the S_n-equivariant morphism $[n]$, cf. ¶2.2.5.

6.2.10. — Ad (5): Our desired S_n-equivariant *simple* semi-homogeneous vector bundle $\mathcal{E}$ will be a graded pieces of a Jordan–Hölder filtration of $\mathcal{F}$. But we face the problem that the equivariant structure of $\mathcal{F}$ does not readily restrict to one of its graded pieces. Studying $\mathcal{F}$ and its Jordan–Hölder filtrations leads to the following information (Proposition 6.2.13).

6.2.11. Situation. — From now on we use the abbreviation

$$X := (A_0 \otimes \Gamma_n) \times (A_0^\vee \otimes \Gamma_n).$$

So we have $g := \dim(X) = 4(n-1)$, and recall that we have $\#X[n] = n^{2g}$.

6.2.12. — Recall Mukai's groups $\Phi_\mu = \operatorname{im}((n, \varphi_{\mathcal{L}})\colon X \to X \times X^\vee)$ as well as $\Sigma_\mu = \ker(\mathrm{pr}_1\colon \Phi_\mu \to X)$ from Definition 1.3.4, where $\mu = \frac{1}{n}[\mathcal{L}]$ and pr_1 is the restriction to Φ_μ of the first coordinate projection. As in ¶1.3.6, we can view Σ_μ as a subgroup of $X^\vee$, and as such it is described as the image

$$\Sigma_\mu = \varphi_{\mathcal{L}}(X[n]) \subset X^\vee[n].$$

6.2.13. Proposition. — *The semi-homogeneous vector bundle $\mathcal{F}$ of slope μ constructed in ¶6.2.9 admits a split Jordan–Hölder filtration, explicitly,*

$$\mathcal{F} \simeq \bigoplus_{\alpha \in X^\vee[n]/\Sigma_\mu} (\mathcal{E}_0 \otimes \mathcal{P}_\alpha)^{\oplus n^{2n-4}},$$

where $\mathcal{E}_0$ is a simple semi-homogeneous vector bundle of slope μ.

We need a bit of preparation before proving Proposition 6.2.13 just after Remark 6.2.17. In particular, we need to calculate the rank of the vector bundles $\mathcal{E}_0$.

6.2.14. — As explained in ¶1.3.14, we can rearrange a Jordan–Hölder filtration of $\mathcal{F}$ into the form

$$\mathcal{F} \simeq \bigoplus_{j \in J} \mathcal{U}_j \otimes \mathcal{E}_j$$

where the simple semi-homogeneous vector bundles $\mathcal{E}_j$ of slope μ are pairwise distinct, and the $\mathcal{U}_j$ are unipotent vector bundles, cf. Definition 1.3.11.

6.2.15. Proposition. — *The group $X^\vee[n]$ acts on the set (of isomorphism classes) $\{\mathcal{E}_j\}_{j \in J}$ via*

$$\alpha.\mathcal{E}_j := \mathcal{E}_j \otimes \mathcal{P}_\alpha,$$

where $\mathcal{P}_\alpha \in \mathrm{Pic}^0(X)$ is the associated line bundle to $\alpha \in X^\vee[n]$. The stabilizer of the action is

$$\mathrm{Stab}(\mathcal{E}_j) = \Sigma_\mu.$$

Proof. — We have by the equality $[n]^*\mathcal{P}_\alpha \simeq \mathcal{P}_{n\alpha} \simeq \mathcal{P}_0 \simeq \mathcal{O}_X$ and the projection formula that

$$\mathcal{F} \otimes \mathcal{P}_\alpha = ([n]_*\mathcal{L}^{\otimes n}) \otimes \mathcal{P}_\alpha \simeq [n]_*(\mathcal{L}^{\otimes n} \otimes [n]^*\mathcal{P}_\alpha) \simeq [n]_*(\mathcal{L}^{\otimes n}) = \mathcal{F}.$$

Recalling that the multiset of associated graded pieces of a Jordan–Hölder filtration is unique, this means that $X^\vee[n]$ acts on the set (of isomorphism classes) $\{\mathcal{E}_j\}_{j\in J}$ via

$$\alpha.\mathcal{E}_j \coloneqq \mathcal{E}_j \otimes \mathcal{P}_\alpha.$$

Next we compute the stabilizers of this action. We can write each $\mathcal{E}_j$ as $\mathcal{E}_0 \otimes \mathcal{P}_{\alpha_j}$ for some $\alpha_j \in X^\vee$ by Proposition 1.3.8, so

$$\operatorname{Stab}(\mathcal{E}_j) = \operatorname{Stab}(\mathcal{E}_0 \otimes \mathcal{P}_{\alpha_j}) = \operatorname{Stab}(\mathcal{E}_0) = \{\alpha \in X^\vee[n] \mid \mathcal{E}_0 \simeq \mathcal{E}_0 \otimes \mathcal{P}_\alpha\}.$$

By ¶1.3.6 this is nothing else than $\Sigma_\mu \cap X^\vee[n] = \Sigma_\mu$, cf. Definition 1.3.4. □

6.2.16. Proposition. — *Let $\mu \in \mathrm{NS}(X) \otimes \mathbb{Q}$ as constructed in ¶6.2.8, and let $\mathcal{E}$ be a simple semi-homogeneous vector bundle on X of slope μ. Then the rank of $\mathcal{E}$ is*

$$\operatorname{rk}(\mathcal{E}) = n^{2n-4}, \quad \textit{and} \quad \#\Sigma_\mu = n^{4n-8}.$$

Proof. — By Proposition 1.3.8 we have $\operatorname{rk}(\mathcal{E})^2 = \deg(\mathrm{pr}_1)$. Now from the equation

$$\Sigma_\mu = \ker(\mathrm{pr}_1) = \{(a,\alpha) \in \Phi_\mu \mid a = 0\} = \{(nx, \varphi_{\mathcal{L}}(x)) \mid nx = 0, x \in X\}$$

we get a short exact sequence

$$0 \to X[n] \cap \ker(\varphi_{\mathcal{L}}) \to X[n] \xrightarrow{\varphi_{\mathcal{L}}} \Sigma_\mu \to 0,$$

and $\operatorname{rk}(\mathcal{E})^2 = \#\varphi_{\mathcal{L}}(X[n])$.

Regarding the kernel, let $(a,\alpha) \in X[n] = ((A_0 \otimes \Gamma_n) \times (A_0^\vee \otimes \Gamma_n))[n]$, then the condition $\varphi_{\mathcal{L}}(a,\alpha) = 0$ becomes

$$\begin{cases} \phi_0(\lambda(a)) = \phi_0(\alpha) & (6.2.1) \\ \phi_0(a) = n_4\phi_0(\lambda^{\mathrm{D}}(\alpha)) & (6.2.2) \end{cases}$$

by $\varphi_{\mathcal{L}} = ng$ and the definition of g. Here we have, by abuse of notation, implicitly applied the maps λ and λ^{D} entry-wise to tuples. Define $\alpha' \coloneqq \lambda(a)$. Since

$$\ker(\phi_0|A_0^\vee \otimes \Gamma_n) = \Delta(A_0^\vee[n])$$

we can write $\alpha = \alpha' + \Delta(\alpha_0)$ for some $\alpha_0 \in A_0^\vee[n]$ by (6.2.1). Similarly, we see after substituting into (6.2.2) that

$$n_4\lambda^{\mathrm{D}}(\alpha' + \Delta(\alpha_0)) = a + \Delta(a_0)$$

for some $a_0 \in A_0[n]$. After substituting $\alpha' = \lambda(a)$, the left hand side of this equals

$$n_4\mathrm{e}(\lambda)a + n_4\lambda^{\mathrm{D}}(\Delta(\alpha_0)) = n_4\mathrm{e}(\lambda)a + \Delta(n_4\lambda^{\mathrm{D}}(\alpha_0)).$$

Using $n_4\mathrm{e}(\lambda) - 1 = n_3 n$ we arrive at

$$0 = n_3 na = \Delta(a_0 - n_4\lambda^{\mathrm{D}}(\alpha_0)),$$

which always admits a solution $a_0 \in A_0[n]$. We conclude that

$$\ker(\varphi_{\mathcal{L}}) \cap X[n] = (A_0 \otimes \Gamma_n)[n] \times \Delta(A_0^\vee[n]).$$

We can now calculate the cardinalities

$$\#(\ker(\varphi_{\mathcal{L}}) \cap X[n]) = n^{4(n-1)} \cdot n^4 = n^{4n},$$

so that we finally get

$$\#\varphi_{\mathcal{L}}(X[n]) = n^{8(n-1)}/n^{4n} = n^{4n-8}$$

and $\mathrm{rk}(\mathcal{E}) = n^{2n-4}$. □

6.2.17. Remark. — Note that the rank of $\mathcal{F}$ is

$$\mathrm{rk}(\mathcal{F}) = \mathrm{rk}([n]_*\mathcal{L}^{\otimes n}) = \deg([n]\colon X \to X) = n^{8(n-1)}.$$

In particular, we see that our $\mathcal{F}$ is definitely not simple. Instead we see that the length N of any Jordan–Hölder filtration of $\mathcal{F}$ will be

$$N = n^{8(n-1)}/n^{2n-4} = n^{6n-4}.$$

Further we see that each orbit of the action in Proposition 6.2.15 contains exactly $n^{4n} = n^{8(n-1)}/n^{4n-8}$ elements.

Proof of Proposition 6.2.13. — We aim to apply the splitting criterion for unipotent vector bundles from Proposition 1.3.12. For this we calculate $\dim \mathrm{End}(\mathcal{F})$ in two ways: First, we abbreviate

$$\widetilde{\mathcal{L}} := \mathcal{L}^{\otimes n}$$

where $\mathcal{L}$ is as in the construction of $\mathcal{F}$. Then we have using ¶1.2.14 that

$$\mathrm{Hom}(\mathcal{F},\mathcal{F}) \simeq \mathrm{Hom}([n]_*\widetilde{\mathcal{L}}, [n]_*\widetilde{\mathcal{L}}) \simeq \mathrm{Hom}([n]^*[n]_*\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}) \simeq \mathrm{Hom}(\bigoplus_{x\in X[n]} \mathrm{t}_x^*\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}),$$

and for $x \in X[n]$ we calculate

$$\mathrm{Hom}(\mathrm{t}_x^*\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}) \simeq \mathrm{Hom}(\widetilde{\mathcal{L}} \otimes \mathcal{P}_{\varphi_{\widetilde{\mathcal{L}}}(x)}, \widetilde{\mathcal{L}}) \simeq \mathrm{Hom}(\mathcal{P}_{\varphi_{\widetilde{\mathcal{L}}}(x)}, \mathcal{O}_X) \simeq \mathrm{H}^0(X, \mathcal{O}_X) \simeq \Bbbk,$$

since $\varphi_{\widetilde{\mathcal{L}}} = n\varphi_{\mathcal{L}}$ implies $\varphi_{\widetilde{\mathcal{L}}}(x) = \varphi_{\mathcal{L}}(nx) = \varphi_{\mathcal{L}}(0) = 0$. We conclude that

$$\dim \mathrm{End}(\mathcal{F}) = \#X[n] = n^{8(n-1)}.$$

Second, we take on the viewpoint that $\mathcal{F} \simeq \bigoplus \mathcal{U}_j \otimes \mathcal{E}_j$. Using $\mathrm{Hom}(\mathcal{E}_j, \mathcal{E}_j) = \Bbbk$ and $\mathrm{Hom}(\mathcal{E}_j, \mathcal{E}_{j'}) = 0$ for $j \neq j'$, cf. ¶1.3.10, we get that

$$\dim \mathrm{End}(\mathcal{F}) = \sum_{j\in J} \dim \mathrm{End}(\mathcal{U}_j).$$

Define $r_j := \mathrm{rk}(\mathcal{U}_j)$ and keep Remark 6.2.17 in the following calculations in mind. By Proposition 6.2.16 we have

$$n^{8(n-1)} = \mathrm{rk}(\mathcal{F}) = \sum_j r_j \,\mathrm{rk}(\mathcal{E}_j) = n^{2n-4} \sum_j r_j,$$

which implies that the r_j's give a partition of $N = n^{6n-4}$. Now we want to take the action from Proposition 6.2.15 into account. Pick a representative $\mathcal{E}_j$ from each orbit and denote the set of indices of these elements by $J_0 \subset J$. Thus the r_j's for $j \in J_0$ constitute a partition of

$$N/(\#\text{elements in an orbit}) = n^{6n-4}/n^{4n} = n^{2n-4}.$$

Making the abbreviation $e_j := \dim \operatorname{End}(\mathcal{U}_j)$, we get from the considerations above that

$$\dim \operatorname{End}(\mathcal{F}) = n^{4n} \sum_{j \in J_0} e_j$$

by Remark 6.2.17 and the fact that the unipotent bundles in an orbit must be isomorphic, cf. ¶1.3.13.

Finally, comparing both dimension computations and using Proposition 1.3.12, we calculate that

$$n^{4n-8} = \sum_{j \in J_0} e_j \leq \sum_{j \in J_0} (r_j)^2 \leq (\sum_{j \in J_0} r_j)^2 = (n^{2n-4})^2.$$

We see that the inequalities have to be equalities, which forces J_0 to be a singleton, say for notation $J_0 = \{0\}$, and also $e_0 = r_0^2$. By Proposition 1.3.12 this means that

$$\mathcal{U}_0 \simeq \mathcal{O}_X^{\oplus n^{2n-4}}$$

as desired. Note that $J_0 = \{0\}$ means that the action of Proposition 6.2.15 is transitive, since by definition J_0 indexes the set of orbits. $\square$

6.2.18. — Now we study the interaction between the S_n-action on X and the split Jordan–Hölder filtration from Proposition 6.2.13. We already saw that $\mathcal{F}$ is S_n-invariant, so pullback along $\sigma \in \mathrm{S}_n$ permutes the graded pieces $\mathcal{E}_j$ of a Jordan–Hölder filtration of $\mathcal{F}$. Taking Proposition 6.2.13 into account, we see that for each $\sigma \in \mathrm{S}_n$ there is a unique

$$\alpha(\sigma) \in X^\vee[n]/\Sigma_\mu \quad \text{such that} \quad \sigma^*\mathcal{E}_0 \simeq \mathcal{E}_0 \otimes \mathcal{P}_{\alpha(\sigma)}.$$

This defines a map

$$\alpha\colon \mathrm{S}_n \to X^\vee[n]/\Sigma_\mu.$$

6.2.19. Proposition. —

(i) *The map $\alpha\colon \mathrm{S}_n \to X^\vee[n]/\Sigma_\mu$ is a crossed homomorphism for the (right) action of S_n inherited from $X^\vee$, and*

(ii) *α is a principal crossed homomorphism if $n \geq 5$ is odd.*

Proof. — (i) Let $\sigma, \tau \in \mathrm{S}_n$. By calculating

$$\mathcal{E}_0 \otimes \mathcal{P}_{\alpha(\sigma\tau)} \simeq (\sigma\tau)^*\mathcal{E}_0 \simeq \tau^*\sigma^*\mathcal{E}_0 \simeq \tau^*(\mathcal{E}_0 \otimes \mathcal{P}_{\alpha(\sigma)}) \\ \simeq \mathcal{E}_0 \otimes \mathcal{P}_{\alpha(\tau)} \otimes \tau^*\mathcal{P}_{\alpha(\sigma)} \simeq \mathcal{E}_0 \otimes \mathcal{P}_{\alpha(\tau)+\tau^\vee(\alpha(\sigma))}$$

we see that

$$\alpha(\sigma\tau) = \alpha(\tau) + \tau^\vee(\alpha(\sigma)),$$

so α is indeed a crossed homomorphism.

(ii) We know by ¶1.3.6 that

$$\Sigma_\mu = \varphi_{\mathcal{L}}(X[n]) = (\varphi_{\mathcal{L}_0} \otimes \phi_0)(X[n]),$$

and by definition $X = (A_0 \times A_0^\vee) \otimes \Gamma_n$, so we have a cokernel sequence

$$(A_0 \times A_0^\vee)[n] \otimes \Gamma_n \xrightarrow{\varphi_{\mathcal{L}_0} \otimes \phi_0} (A_0 \times A_0^\vee)^\vee[n] \otimes \Gamma_n^\vee \to X^\vee[n]/\Sigma_\mu \to 0.$$

Define $\Sigma_0 := \varphi_{\mathcal{L}_0}((A_0 \times A_0^\vee)[n])$, then we get the exact sequence

$$0 \to \Sigma_0 \xrightarrow{\Delta} \Sigma_0 \otimes \Gamma_n \xrightarrow{\mathrm{id} \otimes \phi_0} (A_0 \times A_0^\vee)^\vee[n] \otimes \Gamma_n^\vee \to X^\vee[n]/\Sigma_\mu \to 0,$$

where we used for exactness at $\Sigma_0 \otimes \Gamma_n$ that

$$\ker(\mathrm{id} \otimes \phi_0) = \{(a, \dots, a) \in \Sigma_0^n \mid \textstyle\sum_{i=1}^n a = 0\}$$

and that Σ_0 consists of n-torsion elements. Applying group cohomology, we get

$$\begin{aligned}\mathrm{H}^1(\mathrm{S}_n, (A_0 \times A_0^\vee)^\vee[n] \otimes \Gamma_n^\vee) &\to \mathrm{H}^1(\mathrm{S}_n, X^\vee[n]/\Sigma_\mu) \\ &\to \mathrm{H}^2(\mathrm{S}_n, (\Sigma_0 \otimes \Gamma_n)/\Sigma_0) \to \mathrm{H}^2(\mathrm{S}_n, (A_0 \times A_0^\vee)^\vee[n] \otimes \Gamma_n^\vee),\end{aligned}$$

where the first term is zero by Proposition 4.2.8, and the last term is zero for $n \geq 7$ by Proposition 4.2.7 and for $n = 5$ by Proposition 6.2.22. Next, using that Σ_0 is n-torsion, we obtain from Proposition 4.1.8 the exact sequence

$$0 \to \Sigma_0 \xrightarrow{\Delta} \Sigma_0 \otimes \Gamma_n \xrightarrow{\mathrm{id} \otimes \phi_0} \Sigma_0 \otimes \Gamma_n^\vee \xrightarrow{\mathrm{sum}} \Sigma_0 \to 0.$$

Applying group cohomology, we get

$$\mathrm{H}^1(\mathrm{S}_n, \Sigma_0 \otimes \Gamma_n^\vee) \to \mathrm{H}^1(\mathrm{S}_n, \Sigma_0) \to \mathrm{H}^2(\mathrm{S}_n, (\Sigma_0 \otimes \Gamma_n)/\Sigma_0) \to \mathrm{H}^2(\mathrm{S}_n, \Sigma_0 \otimes \Gamma_n^\vee),$$

where the first term is zero for $n \geq 5$ by Proposition 4.2.8, and the last term is zero for $n \geq 7$ by Proposition 4.2.7 and for $n = 5$ by Proposition 6.2.22.

Finally, using that Σ_0 is abelian and has no 2-torsion since n is odd, we see that

$$\mathrm{H}^1(\mathrm{S}_n, \Sigma_0) \simeq \mathrm{Hom}(\mathrm{S}_n, \Sigma_0) \simeq \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \Sigma_0) = 0.$$

In conclusion,

$$\mathrm{H}^1(\mathrm{S}_n, X^\vee[n]/\Sigma_\mu) = 0,$$

so the crossed homomorphism $\alpha \colon \mathrm{S}_n \to X^\vee[n]/\Sigma_\mu$ must be principal. □

6.2.20. Remark. — In certain situations the proof of Corollary 6.2.21.(ii) can be simplified. The concrete definition of $\varphi_{\mathcal{L}_0}$ above lets one calculate that

$$\ker(\varphi_{\mathcal{L}_0}) \simeq A_0[\mathrm{e}(\lambda) - 1].$$

So if in addition to $\gcd(\mathrm{e}(\lambda), n) = 1$ also $\gcd(\mathrm{e}(\lambda) - 1, n) = 1$ holds, we get that the homomorphism $\varphi_{\mathcal{L}_0} \colon (A_0 \times A_0^\vee)[n] \to (A_0 \times A_0^\vee)^\vee[n]$ is an isomorphism. Then we see that

$$X^\vee[n]/\Sigma_\mu = ((A_0 \times A_0^\vee)^\vee[n] \otimes \Gamma_n^\vee)/(\varphi_{\mathcal{L}_0} \otimes \phi_0)((A_0 \times A_0^\vee)[n] \otimes \Gamma_n)$$

actually carries a trivial S_n-action, since S_n acts trivially on $\Gamma_n^\vee/\phi_0(\Gamma_n)$ by Proposition 4.1.7. This renders most of the proof of (ii) in Corollary 6.2.21 unnecessary.

6.2.21. Corollary. — *Assume that $n \geq 5$ is odd. Then there exists $\alpha_0 \in X^\vee[n]/\Sigma_\mu$ such that $\mathcal{E}_0 \otimes \mathcal{P}_{\alpha_0}$ is S_n-invariant.*

Proof. — We know that the crossed homomorphism $\alpha\colon \mathrm{S}_n \to X^\vee[n]/\Sigma_\mu$ must be principal by Proposition 6.2.19, i.e. there exists some $\alpha_0 \in X^\vee[n]/\Sigma_\mu$ such that $\alpha(\sigma) = \sigma^\vee(\alpha_0) - \alpha_0$ for each $\sigma \in \mathrm{S}_n$. This means by construction of α that $\mathcal{E}_0 \otimes \mathcal{P}_{-\alpha_0}$ is S_n-invariant. □

We used the following proposition in the proof of Proposition 6.2.19.

6.2.22. Proposition. — *We have $\mathrm{H}^2(\mathrm{S}_5, A[5] \otimes \Gamma_5^\vee) = 0$ and $\mathrm{H}^2(\mathrm{S}_5, \Sigma_0 \otimes \Gamma_5^\vee) = 0$ where A is an abelian variety and Σ_0 is the group from above.*

Proof. — We claim that $\mathrm{H}^3(\mathrm{S}_5, \mathbb{Z}/5\mathbb{Z})$ vanishes. Apply group cohomology to the sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 5} \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z} \to 0$$

to get the exact sequence

$$\mathrm{H}^3(\mathrm{S}_5, \mathbb{Z}) \xrightarrow{\cdot 5} \mathrm{H}^3(\mathrm{S}_5, \mathbb{Z}) \to \mathrm{H}^3(\mathrm{S}_5, \mathbb{Z}/5\mathbb{Z}) \to \mathrm{H}^4(\mathrm{S}_5, \mathbb{Z}) \xrightarrow{\cdot 5} \mathrm{H}^4(\mathrm{S}_5, \mathbb{Z})$$

We know from Table 1 on page 63 that $\mathrm{H}^4(\mathrm{S}_5, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $\mathrm{H}^3(\mathrm{S}_5, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, so multiplication by 5 is an isomorphism on both.

By Proposition 4.2.7 we know that $\mathrm{H}^2(\mathrm{S}_5, A[5] \otimes \Gamma_5^\vee)$ injects into $\mathrm{H}^3(\mathrm{S}_5, A[5])$, but $A[5]$ is isomorphic to a direct sum of copies of $\mathbb{Z}/5\mathbb{Z}$. Similarly, the group Σ_0 is a finite abelian group which is 5-torsion, so it is also isomorphic to a direct sum of copies of $\mathbb{Z}/5\mathbb{Z}$, so we can again conclude by Proposition 4.2.7. □

Finally we can use the information gained above to prove Theorem 6.2.1.

Proof of Theorem 6.2.1. — To summarize so far, we have constructed an S_n-equivariant sheaf $\mathcal{F}$ following the steps of Construction 6.2.3, where we have executed these steps concretely in ¶¶6.2.4, 6.2.5, 6.2.8 and 6.2.9 while carrying an S_n-equivariant structure along; the sheaf $\mathcal{F}$ itself is defined in ¶6.2.9. In Proposition 6.2.13 we have seen that $\mathcal{F}$ is the direct sum

$$\mathcal{F} \simeq \bigoplus_\alpha (\mathcal{E}_0 \otimes \mathcal{P}_\alpha)^{\oplus r}$$

of simple semi-homogeneous vector bundles, where $\alpha \in X^\vee[n]/\Sigma_\mu$ and $r = n^{2n-4}$, and each of the summands provides the kernel of a derived equivalence

$$\mathrm{FM}_{\mathcal{E}_0 \otimes \mathcal{P}_\alpha}\colon \mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(A^\vee \otimes \Gamma_n).$$

For $n \geq 5$ odd, at least one of these summands must be S_n-invariant by Corollary 6.2.21, say $\mathcal{E} := \mathcal{E}_0 \otimes \mathcal{P}_{\alpha_0}$. Since the sheaves $\mathcal{E}_0 \otimes \mathcal{P}_\alpha$ are pairwise non-isomorphic, there are no non-zero homomorphisms between them, cf. ¶1.3.10, so the S_n-equivariant structure on $\mathcal{F}$ restricts to one on $\mathcal{E}^{\oplus r}$. Also recall that $r = n^{2n-4}$ is odd since n is odd. Now Propositions 2.2.15 and 2.2.17 explain that the S_n-invariant sheaf $\mathcal{E}$ inherits an S_n-equivariant structure from $\mathcal{E}^{\oplus r}$. Here we used again that r is odd.

These arguments culminate in the construction of an S_n-equivariant kernel

$$(\mathcal{E}, \phi) \in \mathbf{D}^{\mathrm{b}}_{\Delta \mathrm{S}_n}((A \otimes \Gamma_n) \times (A^\vee \otimes \Gamma_n))$$

such that $\mathrm{FM}_{\mathcal{E}} \colon \mathbf{D}^{\mathrm{b}}(A \otimes \Gamma_n) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(A^\vee \otimes \Gamma_n)$ is a derived equivalence. So we can conclude exactly as in the proof of Theorem 6.1.8, using Ploog's method (cf. §2.2) and the equivariant viewpoint on derived categories of generalized Kummer varieties (cf. Proposition 2.2.21). $\square$

6.2.23. Remark. — Note that the same techniques that lead to derived equivalences of generalized Kummer varieties associated to A and $A^\vee$ can in principle be applied to any pair of abelian varieties A and B, as long as one has sufficient information about the homomorphisms between A and B.

Part III

Miscellaneous results

CHAPTER 7

Birationality of generalized Kummer varieties

In this chapter we study birational equivalences between generalized Kummer varieties with the goal to exhibit examples which are not birationally equivalent but are derived equivalent according to Theorem 1. Before this, we recall for context some results which establish isomorphisms between generalized Kummer varieties associated to an abelian variety and its dual. After the expositional part at the beginning of this section, the calculations that lead to Theorems 7.2.8 and 7.2.12 are original. The interested reader is invited to consult the literature [ADM16; MMY20; Oka21] for further examples of non-birational but derived equivalent hyperkähler varieties.

7.0.1. Situation. — In this chapter we work over the complex numbers $\mathbb{C}$, in order to consider singular cohomology of (analytifications of) varieties. Let A be an abelian surface over $\mathbb{C}$, and let $\lambda\colon A \to A^\vee$ be some polarization of A of degree $\deg(\lambda) = d^2$.

7.1. Isomorphic generalized Kummer varieties

7.1.1. Proposition. — *We always have an isomorphism* $\mathrm{Kum}^1(A) \simeq \mathrm{Kum}^1(A^\vee)$ *of Kummer surfaces.*

Proof. — By Mukai [Muk81], we know that A and $A^\vee$ are derived equivalent, i.e. $\mathbf{D}^{\mathrm{b}}(A) \simeq \mathbf{D}^{\mathrm{b}}(A^\vee)$. So by [HLOY03, Thm. 0.1] (recalled as Theorem 2.3.16), it follows that $\mathrm{Kum}^1(A)$ and $\mathrm{Kum}^1(A^\vee)$ are isomorphic. The proof of the latter theorem relies on a comparison of transcendental lattices, so we want to discuss a lattice theoretic argument which sidesteps the use of derived equivalences. Following [HuyK3, §3.2.5], the construction of the Kummer surface $\mathrm{Kum}^1(A)$, cf. Example 1.1.6, as a quotient of the blowup $\widetilde{A}$ of A in the two-torsion points $A[2] \subset A$ leads to a Hodge isometry

$$\mathrm{H}^2(\widetilde{A}, \mathbb{Z}) \simeq \mathrm{H}^2(A, \mathbb{Z}) \oplus \bigoplus_{i=1}^{16} \mathbb{Z} \cdot [E_i]$$

and a Hodge sublattice

$$\mathrm{H}^2(\mathrm{Kum}^1(A), \mathbb{Z}) \supset \mathrm{H}^2(A, \mathbb{Z}) \oplus \bigoplus_{i=1}^{16} \mathbb{Z} \cdot [\overline{E_i}], \tag{7.1.1}$$

where $E_i \subset \widetilde{A}$ denote the exceptional divisors and $\overline{E_i}$ their image under the quotient projection. Since one has a Hodge-isometry

$$\mathrm{H}^2(A,\mathbb{Z}) \overset{\text{cup}}{\simeq} \mathrm{H}^2(A,\mathbb{Z})^\vee \simeq \wedge^2 \mathrm{H}^1(A,\mathbb{Z})^\vee \overset{\text{cap}}{\simeq} \wedge^2 \mathrm{H}_1(A,\mathbb{Z}) \simeq \wedge^2 \mathrm{H}^1(A^\vee,\mathbb{Z}) \simeq \mathrm{H}^2(A^\vee,\mathbb{Z}),$$

cf. [Nam02a, ¶3] and [Shi78, Lem. 3], one would like to extend this isometry to one between $\mathrm{H}^2(\mathrm{Kum}^1(A),\mathbb{Z})$ and $\mathrm{H}^2(\mathrm{Kum}^1(A^\vee),\mathbb{Z})$ and conclude using the Torelli theorem for K3 surfaces [HuyK3, Thm. 3.2.4]. But even when saturating the right summand in (7.1.1) to get the Kummer lattice K, one still has a proper inclusion

$$\mathrm{H}^2(A,\mathbb{Z}) \oplus \mathrm{K} \subset \mathrm{H}^2(\mathrm{Kum}^1(A),\mathbb{Z}),$$

and one has trouble to extend the Hodge isometry, cf. [Nam02a, Rmk. 1]. Instead [HLOY03] focus on the *transcendental lattices* $\mathrm{T}(A) \coloneqq \mathrm{NS}(A)^\perp \subset \mathrm{H}^2(A,\mathbb{Z})$ and $\mathrm{T}(\mathrm{Kum}^1(A)) \coloneqq \mathrm{NS}(\mathrm{Kum}^1(A))^\perp \subset \mathrm{H}^2(\mathrm{Kum}^1(A),\mathbb{Z})$, where one has

$$\mathrm{T}(\mathrm{Kum}^1(A)) \simeq \mathrm{T}(A)(2),$$

cf. [Nik76, Eq. (6)] and [Mor84, Lem. 3.1]; here the notation "(2)" signifies that the pairing of the lattice is scaled by the factor 2. So the isometry $\mathrm{T}(A) \simeq \mathrm{T}(A^\vee)$ induces an isometry

$$\mathrm{T}(\mathrm{Kum}^1(A)) \simeq \mathrm{T}(\mathrm{Kum}^1(A^\vee)),$$

which must come from an isomorphism $\mathrm{Kum}^1(A) \simeq \mathrm{Kum}^1(A^\vee)$ by [Muk87, Prop. 6.2]. □

7.1.2. Proposition. — *In the case $d = 1$, we have $\mathrm{Kum}^m(A) \simeq \mathrm{Kum}^m(A^\vee)$ for any integer $m \geq 1$.*

Proof. — The hypothesis $d = 1$ means that $\lambda\colon A \to A^\vee$ is an isomorphism, i.e. A is principally polarizable. As the Hilbert scheme of points on A and the Hilbert–Chow morphism respect isomorphisms, the construction of generalized Kummer varieties respects isomorphisms as well. □

7.1.3. Proposition. — *Assume that $\mathrm{End}(A) = \mathbb{Z}$ and that $\lambda\colon A \to A^\vee$ is the polarization of minimal degree $\deg(\lambda) = d^2$. Set $n \coloneqq d + 1$. Then we have isomorphisms*

(i) *$\mathrm{Hilb}^n(A) \times A^\vee \simeq \mathrm{Hilb}^n(A^\vee) \times A^{\vee\vee}$, and*
(ii) *$\mathrm{Kum}^d(A) \simeq \mathrm{Kum}^d(A^\vee)$.*

Proof. — The polarization λ corresponds to an ample class $h \in \mathrm{NS}(A) \subset \mathrm{H}^2(A,\mathbb{Z})$ of degree $h^2 = 2d$, i.e. $\lambda = \varphi_{\mathcal{L}}$ with $h = [\mathcal{L}]$, cf. Remark 1.2.18. We will apply Yoshioka's results [Yos01] on moduli spaces $\mathrm{M}_{(A,h)}(\nu)$ of stable sheaves on abelian surfaces A; see loc. cit. and [HL] for background on this theory.

(i) Consider the Mukai vector

$$\nu_1 \coloneqq 1 + 0 - n\omega \in \widetilde{\mathrm{H}}(A,\mathbb{Z}) \coloneqq \mathrm{H}^0(A,\mathbb{Z}) \oplus \mathrm{H}^2(A,\mathbb{Z}) \oplus \mathrm{H}^4(A,\mathbb{Z}),$$

where $\mathrm{H}^4(A,\mathbb{Z}) = \mathbb{Z}\cdot\omega$ is spanned by the fundamental class ω. Denote by $\mathrm{M}_{(A,h)}(\nu_1)$ the moduli space of h-Gieseker stable sheaves $\mathcal{F}$ on A with Mukai vector $\nu(\mathcal{F}) = \nu_1$.

By [Yos01, §4.3.1] we have an isomorphism

$$\mathrm{M}_{(A,h)}(\nu_1) \simeq \mathrm{Hilb}^n(A) \times A^\vee. \tag{7.1.2}$$

Next, consider the Mukai vector $\nu_2 := 1 + h + (d-n)\omega \in \widetilde{\mathrm{H}}(A,\mathbb{Z})$. Then we have $\nu_2^2 = h^2 - 2(d-n) = 2n$, computed in the Mukai lattice $\widetilde{\mathrm{H}}(A,\mathbb{Z})$, so

$$\begin{aligned}\nu_1 \cdot \exp(h) &= (1 - n\omega) \cdot (1 + h + (\tfrac{1}{2}h^2)\omega)\\ &= 1 + h + (\tfrac{1}{2}h^2)\omega - n\omega\\ &= 1 + h + (\tfrac{1}{2}h^2)\omega - (\tfrac{1}{2}h^2 - (d-n))\omega\\ &= 1 + h + (d-n)\omega\\ &= \nu_2.\end{aligned}$$

By [Yos01, Lem. 1.1] we have

$$\mathrm{M}_{(A,h)}(\nu_1) \simeq \mathrm{M}_{(A,h)}(\nu_1 \cdot \exp(h)) = \mathrm{M}_{(A,h)}(\nu_2), \tag{7.1.3}$$

since $\mathrm{rk}(\nu_1) = 1 > 0$ and h is an algebraic class. Now we consider the dual abelian variety $A^\vee$ and recall that $\mathrm{End}(A) = \mathbb{Z}$ implies $\mathrm{NS}(A^\vee) = \mathbb{Z} \cdot \check{h}$, where $\check{h}$ denotes the ample class corresponding to the dual polarization $\lambda^\delta \colon A^\vee \to A$, which has again degree $\deg(\lambda^\delta) = d^2$ with $2d = \check{h}^2$, cf. Propositions 1.2.25 and 1.2.27. Thus, by [Yos01, Prop. 3.5], we obtain an isomorphism

$$\mathrm{M}_{(A,h)}(1 + h - \omega) \simeq \mathrm{M}_{(A^\vee,\check{h})}(1 + \check{h} - \check{\omega}). \tag{7.1.4}$$

The desired isomorphism in statement (i) is now a composition of the isomorphisms in (7.1.2), (7.1.3), and (7.1.4).

(ii) Since the generalized Kummer variety $\mathrm{Kum}^{n-1}(A)$ is the fiber of the Albanese map

$$\mathrm{Hilb}^n(A) \times A^\vee \to A \times A^\vee$$

over $(0,0) \in A \times A^\vee$, cf. [Yos01, §4.3.1], and Albanese maps are unique up to a unique isomorphism of abelian varieties by their universal property, cf. [BeaCAS, Thm. V.13], the isomorphism constructed in (i) restricts to an isomorphism

$$\mathrm{Kum}^{n-1}(A) \simeq \mathrm{Kum}^{n-1}(A^\vee).$$

In fact, Yoshioka describes the Albanese map $\mathrm{alb}_{A,\nu} \colon \mathrm{M}_{(A,h)}(\nu) \to A \times A^\vee$ concretely in [Yos01, §4.1]. By inspection of the definition, the isomorphism (7.1.3) is seen to be compatible with the Albanese maps alb_{A,ν_1} and $\mathrm{alb}_{A,\nu_1 \exp(h)}$.[(1)] The compatibility of (7.1.4) with the Albanese maps alb_{A,ν_2} and $\mathrm{alb}_{A^\vee,\check{\nu}_2}$ is explained in [Yos01, Prop. 4.9].[(2)] □

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(1) When inspecting the definition, one has to recall that $\det(\mathcal{E} \otimes \mathcal{L}) \simeq \det(\mathcal{E}) \otimes \mathcal{L}^{\otimes \mathrm{rk}(\mathcal{E})}$ for each $\mathcal{E} \in \mathrm{M}_{(A,h)}(\nu)$ and $\mathcal{L} \in \mathrm{Pic}(A)$, and that $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\nu)$ is independent of $\mathcal{E}$ since it is determined by the Mukai vector ν.

(2) Here one has to realize that the moduli space of simple semi-homogeneous vector bundles $\mathcal{F}$ on A with $\mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\mathcal{O}(h))$, as occurring in [Yos01, §4.2], is isomorphic to $A^\vee$.

7.2. Non-birational generalized Kummer varieties

***7.2.1. Theorem* (Namikawa [Nam02a]).** — *Assume that* $\mathrm{End}(A) = \mathbb{Z}$*, and that the polarization* $\lambda\colon A \to A^\vee$ *of minimal degree satisfies* $\deg(\lambda) = 3^2$*. Then the generalized Kummer fourfolds* $\mathrm{Kum}^2(A)$ *and* $\mathrm{Kum}^2(A^\vee)$ *are* not *birationally equivalent.*

We want to reproduce the proof of this theorem, since below we will build directly upon it. In order to do so, we need to recall a few facts about generalized Kummer varieties, cf. Chapter 1. These facts are mentioned and used in [Nam02a, ¶¶1–4] and [Oka21, Prop. 2.2]; we spell out some explanations.

7.2.2. — We consider generalized Kummer varieties of dimension 4.

(i) Specializing ¶1.1.18.(ii) to the case $n := 3$, we have a Hodge isometry

$$\mathrm{H}^2(\mathrm{Kum}^2(A), \mathbb{Z}) \simeq \mathrm{H}^2(A, \mathbb{Z}) \oplus \mathbb{Z}\delta, \tag{7.2.1}$$

where on the right hand side $\delta^2 = -6$, and $2\delta = [E]$ is represented by the exceptional divisor $E \subset \mathrm{Kum}^2(A)$ associated to the Hilbert–Chow morphism. See ¶7.2.3 for more information about E. One can find a geometric discussion of δ and E in [Bea83, §§6–7] and [Yos01, §4.3.1], for example.

(ii) In particular, we obtain from the Lefschetz theorem on $(1,1)$-classes and (7.2.1) an isomorphism

$$\mathrm{NS}(\mathrm{Kum}^2(A)) \simeq \mathrm{NS}(A) \oplus \mathbb{Z}\delta.$$

In fact, the first Chern class $\mathrm{c}_1\colon \mathrm{Pic}(\mathrm{Kum}^2(A)) \to \mathrm{NS}(\mathrm{Kum}^2(A))$ is observed to be an isomorphism by studying the exponential sequence, since by simply-connectedness of $\mathrm{Kum}^2(A)$, we have that $\mathrm{H}^1(\mathrm{Kum}^2(A), \mathbb{Z}) = 0$, which implies $\mathrm{H}^1(\mathrm{Kum}^2(A), \mathcal{O}) = 0$ by the Hodge decomposition. So

$$\mathrm{Pic}(\mathrm{Kum}^2(A)) \simeq \mathrm{NS}(A) \oplus \mathbb{Z}\delta.$$

(iii) Let us take a lattice theoretic viewpoint, see [HuyK3, Ch. 14] for an introduction. The lattice

$$\mathrm{U} := \mathbb{Z} \cdot \mathrm{e} \oplus \mathbb{Z} \cdot \mathrm{f}$$

with $\mathrm{e}^2 = 0$, $\mathrm{f}^2 = 0$, and $\mathrm{e}.\mathrm{f} = 1$ is called the *hyperbolic plane*. For $k \in \mathbb{Z}$, denote by $\langle k \rangle := \mathbb{Z} \cdot g$ the rank 1 lattice with $g^2 = k$.

By Remark 1.2.5 we have $\mathrm{H}^2(A, \mathbb{Z}) \simeq \wedge^2(\mathbb{Z}^{\oplus 4})$, compatible with cup-product on the left hand side and exterior product on the right hand side. So one verifies straightforwardly that one has isometries

$$\mathrm{H}^2(A, \mathbb{Z}) \simeq \mathrm{U}^{\oplus 3} \tag{7.2.2}$$

and

$$\mathrm{H}^2(\mathrm{Kum}^2(A), \mathbb{Z}) \simeq \mathrm{U}^{\oplus 3} \oplus \langle -6 \rangle.$$

If $h \in \mathrm{NS}(A)$ is a primitive class, i.e. it is not a multiple of any other class, with $h^2 = 2d$, then one can choose the isometry (7.2.2) in such a way that h corresponds to $\mathrm{e} + d\mathrm{f}$ in the first copy of U inside $\mathrm{U}^{\oplus 3}$, cf. [HuyK3, Cor. 14.1.10].

7.2.3. — Denote by $E \subset \mathrm{Kum}^2(A)$ the exceptional divisor of the desingularization given by the Hilbert–Chow morphism $\mathrm{HC}\colon \mathrm{Kum}^2(A) \to (A \otimes \Gamma_3)/\mathrm{S}_3$.

(i) The exceptional divisor $E \subset \mathrm{Kum}^2(A)$ is effective and *rigid*, i.e.

$$\dim \mathrm{H}^0(\mathrm{Kum}^2(A), \mathcal{O}(E)) = 1,$$

since it is the exceptional divisor of a desingularization of a normal, projective variety. Since we have $\mathrm{H}^1(\mathrm{Kum}^2(A), \mathcal{O}) = 0$, this is equivalent to the vanishing of $\mathrm{H}^0(\mathrm{Kum}^2(A), \mathcal{N}_{E/\mathrm{Kum}^2(A)}) = 0$, where $\mathcal{N}_{E/\mathrm{Kum}^2(A)} \simeq \mathcal{O}_E(E)$ is the normal bundle of $E \subset \mathrm{Kum}^2(A)$.

(ii) Let $\widetilde{E} \to E$ be a desingularization of E. Then the Albanese of $\widetilde{E}$ is

$$\mathrm{Alb}(\widetilde{E}) \simeq A.$$

See [Ser59] and [BeaCAS, Thm. V.13] as references on Albanese varieties. Indeed, the singular locus of $(A \otimes \Gamma_3)/\mathrm{S}_3 \subset \mathrm{Sym}^3(A)$ is given by $\Delta = \{a + a + b \mid a \in A, b = -2a\}$. The morphism $A \to \Delta$ mapping $a \mapsto a + a + (-2a)$ induces an isomorphism $A \setminus A[3] \xrightarrow{\sim} \Delta^\circ$, where

$$\Delta^\circ := \Delta \setminus \{a + a + a \mid a \in A[3]\}.$$

The rational map $\widetilde{E} \to E \to \Delta \dashrightarrow A$ extends to a morphism $\widetilde{E} \to A$ since $\widetilde{E}$ is smooth and A is an abelian variety, cf. [Mil86, Thm. 3.1]. We claim that this is an Albanese morphism (after choosing suitable basepoints, which we elide in the discussion below).

If $x = a+a+b \in \Delta^\circ$, then $\mathrm{HC}^{-1}(\{x\}) \simeq \mathrm{Hilb}^2(A, a) \times \mathrm{Hilb}^1(A, b)$ is a product of local Hilbert schemes, cf. [Iar72, p. 820], [Bri77]. But $\mathrm{Hilb}^1(A, b) \simeq \mathrm{pt}$ is a point and $\mathrm{Hilb}^2(A, a) \simeq \mathbb{P}(\mathrm{T}_{A,a})$ is the projective tangent space of A at a. Writing $E^\circ := \mathrm{HC}^{-1}(\Delta^\circ)$, this leads to an isomorphism $E^\circ \simeq \mathbb{P}(\mathcal{T}_{\Delta^\circ})$, and since $\mathcal{T}_A \simeq \mathcal{O}_A^{\oplus 2}$ is (locally) trivial, we obtain that

$$E^\circ \simeq \Delta^\circ \times \mathbb{P}^1.$$

We consider the universal property describing an Albanese morphism. So let $f\colon \widetilde{E} \to B$ be a morphism to an abelian variety B. This induces a rational map

$$f\colon \Delta^\circ \times \mathbb{P}^1 \simeq E^\circ \dashrightarrow B.$$

But any rational map from a product to an abelian variety decomposes as a sum $f = f_1 + f_2$ of rational maps $f_1\colon \Delta^\circ \dashrightarrow B$ and $f_2\colon \mathbb{P}^1 \dashrightarrow B$, cf. [LanAV, Thm. II.3]. Now since

$$\dim(\mathrm{Alb}(\mathbb{P}^1)) = \dim(\mathrm{H}^0(\mathbb{P}^1, \Omega^1)) = 0,$$

the morphism f_2 must be constant, say $f_2 = 0$ without loss of generality. Finally, since Δ° is birationally equivalent to A and A is smooth, f_1 extends to a morphism $f_1\colon A \to B$, which witnesses the factorization of $f\colon \widetilde{E} \to B$ over the morphism $\widetilde{E} \to A$ (note here that $E^\circ \hookrightarrow \widetilde{E}$ is a dense open subset). Uniqueness of the factorization is easily verified using the density of $\Delta^\circ \hookrightarrow A$.

7.2.4. — Let X and Y be smooth, projective varieties, and let $f\colon X \overset{\sim}{\dashrightarrow} Y$ be a birational map.

(i) Assume that X and Y have trivial canonical bundles $\omega_X \simeq \mathcal{O}_X$ and $\omega_Y \simeq \mathcal{O}_Y$, respectively. Then f is an isomorphism in codimension 1, i.e. there exists open subsets $U \subset X$ and $V \subset Y$ with $\operatorname{codim}_X(X \setminus U) \geq 2$ and $\operatorname{codim}_Y(Y \setminus V) \geq 2$ such that $f\colon U \xrightarrow{\sim} V$ is an isomorphism, cf. [Bat99, Prop. 3.1]. So we obtain an isomorphism
$$f^*\colon \mathrm{H}^2(Y,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(X,\mathbb{Z}), \tag{7.2.3}$$
since $\mathrm{H}^2(X,\mathbb{Z}) \simeq \mathrm{H}^2(U,\mathbb{Z})$ and $\mathrm{H}^2(Y,\mathbb{Z}) \simeq \mathrm{H}^2(V,\mathbb{Z})$, cf. [IveCS, IX.2.1, Thm. IX.4.7].

(ii) If X and Y are hyperkähler varieties, e.g. $X = \mathrm{Kum}^2(A)$ and $Y = \mathrm{Kum}^2(A^\vee)$, then (7.2.3) is a Hodge isometry, where the second cohomology groups are endowed with their Beauville–Bogomolov–Fujiki form, cf. [OGr97, Prop. 1.6.2], [Huy99, Lem. 2.6].

(iii) Since $\mathrm{NS}(X)$ is a finitely generated abelian group, $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is a finite-dimensional vector space over $\mathbb{R}$, which we endow with the euclidean topology. A divisor $D \in \mathrm{Div}(X)$ is called *moveable* if
$$\operatorname{codim}_X(\mathrm{Bl}(|D|)) \geq 2,$$
i.e. the base locus of the complete linear system $|D|$ has codimension at least 2, cf. [Yos16, Def. 1.7]. Denote the convex cone generated by classes of movable divisors by
$$\mathrm{Mov}(X) \subset \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R},$$
and denote it closure in the euclidean topology by $\overline{\mathrm{Mov}}(X)$.

Let $f\colon X \overset{\sim}{\dashrightarrow} Y$ be a birational map which is an isomorphism in codimension 1. We see that if $D \in \mathrm{Div}(Y)$ is moveable, then $f^*D \in \mathrm{Div}(X)$ is also moveable. So the isomorphism $f^*\colon \mathrm{NS}(Y) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ restricts to an isomorphism
$$f^*\colon \overline{\mathrm{Mov}}(Y) \xrightarrow{\sim} \overline{\mathrm{Mov}}(X)$$
of convex cones.

Recall that if C is a convex cone, one calls $\mathbb{R}_{\geq 0} \cdot x \subset C$ an *extremal ray* if for any $x_1, x_2 \in C$ and $\alpha_1, \alpha_2 \in \mathbb{R}_{>0}$ the equation $x = \alpha_1 x_1 + \alpha_2 x_2$ implies that $x_1, x_2 \in \mathbb{R}_{\geq 0} \cdot x$. Note that f^* maps extremal rays to extremal rays.

Proof of Theorem 7.2.1. — We follow the proof from [Nam02a]. By the assumptions on A, we can write $\mathrm{NS}(A) = \mathbb{Z} \cdot h$ with $h^2 = 2d$ and $d := 3$. So, by ¶7.2.2, we have
$$\mathrm{Pic}(\mathrm{Kum}^2(A)) \simeq \mathbb{Z} \cdot h \oplus \mathbb{Z} \cdot \delta,$$
where $\delta^2 = -6$ and $h.\delta = 0$. The analogous statement holds for $A^\vee$, where we write $\check{h}$ and $\check{\delta}$ in place of h and δ.

Let $f\colon \mathrm{Kum}^2(A) \overset{\sim}{\dashrightarrow} \mathrm{Kum}^2(A^\vee)$ be a birational equivalence. By ¶7.2.4, f is an isomorphism in codimension 1 and induces a Hodge-isometry $f^*\colon \mathrm{H}^2(\mathrm{Kum}^2(A^\vee),\mathbb{Z}) \to \mathrm{H}^2(\mathrm{Kum}^2(A),\mathbb{Z})$, and hence it induces an isomorphism of lattices
$$f^*\colon \mathrm{Pic}(\mathrm{Kum}^2(A^\vee)) \xrightarrow{\sim} \mathrm{Pic}(\mathrm{Kum}^2(A)).$$

We assume that $f^*(\check{\delta}) = \pm\delta$, which we verify at the end of the proof. Then in fact

$$f^*(\check{\delta}) = \delta,$$

since 2δ and $2\check{\delta}$ are represented by the effective divisors E and $\check{E}$, cf. ¶7.2.2. Since E is rigid, cf. ¶7.2.3, $f\colon \operatorname{Kum}^2(A) \overset{\sim}{\dashrightarrow} \operatorname{Kum}^2(A^\vee)$ restricts to a birational map

$$f|_E : E \overset{\sim}{\dashrightarrow} \check{E}.$$

Then the desingularizations $\widetilde{E}$ and $\widetilde{\check{E}}$ of E and $\check{E}$, respectively, are also birationally equivalent, and since they are smooth, this induces by ¶7.2.3 an isomorphism of Albanese varieties

$$A \simeq \operatorname{Alb}(\widetilde{E}) \xrightarrow{\sim} \operatorname{Alb}(\widetilde{\check{E}}) \simeq A^\vee.$$

This contradicts the assumption on the minimal degree of a polarization of A.

Finally we verify that indeed $f^*(\check{\delta}) = \delta$; for a more general calculation see the proof of Theorem 7.2.8. Write $f^*(\check{\delta}) = zh + w\delta$ for some $z, w \in \mathbb{Z}$. Since $\check{\delta}^2 = -6$, we obtain the equation

$$-6 = f^*(\check{\delta})^2 = (zh + w\delta)^2 = 6z^2 - 6w^2,$$

which is only satisfied when $w = \pm 1$ and $z = 0$. □

To state and prove our generalization of Namikawa's theorem we need to recall a few facts about Pell equations.

***7.2.5.* Pell equations.** — See [JKPell] for reference, for example. Fix some $N \in \mathbb{N}$. A *Pell equation* is a diophantine equation, i.e. $x, y \in \mathbb{Z}$, of the form

$$x^2 - Ny^2 = 1. \tag{7.2.4}$$

The trivial solutions are $(x, y) = (\pm 1, 0)$ and always exist. If N is a perfect square, then (7.2.4) has only trivial solutions; the converse is in fact also true. In the case that N is not a perfect square, the *fundamental solution* (x_0, y_0) is the solution of (7.2.4) with $x_0, y_0 > 0$ and $|x_0|$ minimal among all solutions (equivalently $|y_0|$ is minimal).

Below we are especially interest in the case $N = 3d$. We list the fundamental solutions of $x^2 - 3dy^2 = 1$, taken from [OEIS, A002349, A002350] in Table 1. The columns with odd y_0 are shaded in gray.

7.2.6. Proposition. — *Consider the Pell equation $x^2 - 3dy^2 = 1$ for $d \in \mathbb{N}$, with fundamental solution (x_0, y_0).*

(i) *There are infinitely many $d \in \mathbb{N}$, with $\gcd(3, d) = 1$, such that $y_0 = 1$.*
(ii) *If x_0 is even, then d and y_0 are odd.*
(iii) *x_0 is even if and only if some solution (x, y) has even x. Equivalently, the equation $4x'^2 - 3dy^2 = 1$ has a solution (here $x = 2x'$).*
(iv) *y_0 is odd if and only if some solution (x, y) has odd y.*

Proof. — (i) For any $k \in \mathbb{N}$ set $d' := 2k^2 + (k+1)^2 - 1$ and $d'' := k^2 + 2(k+1)^2 - 1$. Then one calculates $(3k+1)^2 - 3d' = 1$ and $(3k+2)^2 - 3d'' = 1$, as desired. Reducing modulo 3 one sees that for every k either d' or d'' is coprime to 3.

(ii) This is elementary.

TABLE 1. Fundamental solutions (x_0, y_0) of $x^2 - 3dy^2 = 1$.

d	4	5	7	8	10	11	13	14	16	17	18	19	20	21	22	23	24
$N = 3d$	12	15	21	24	30	33	39	42	48	51	54	57	60	63	66	69	72
x_0	7	4	55	5	11	23	25	13	7	50	485	151	31	8	65	7775	17
y_0	2	1	12	1	2	4	4	2	1	7	66	20	4	1	8	936	2

(iii) For Pell equations in general, the recurrence system

$$\begin{cases} x_{n+1} = x_0 x_n + 3d y_0 y_n & (7.2.5) \\ y_{n+1} = x_0 y_n + y_0 x_n & (7.2.6) \end{cases}$$

enumerates all solutions $(\pm x_n, \pm y_n)_n$. We claim that we have the recurrence relation

$$x_{n+1} = 2x_0 x_n - x_{n-1} \tag{7.2.7}$$

with $x_{-1} \coloneqq 1$. By the recurrence system equations and Pell's equation, we have

$$\begin{aligned} x_{n+1} &= x_0 x_n + 3d y_0 y_n \\ &= x_0 x_n + 3d y_0 (x_0 y_{n-1} + y_0 x_{n-1}) \\ &= x_0 x_n + 3d y_0 x_0 y_{n-1} + 3d y_0^2 x_{n-1} \\ &= x_0 x_n + 3d y_0 x_0 y_{n-1} + (x_0^2 - 1) x_{n-1} \end{aligned}$$

But since (7.2.5) multiplied with x_0 says $x_0 x_n = x_0^2 x_{n-1} + 3d y_0 x_0 y_{n-1}$, we have

$$3d y_0 x_0 y_{n-1} = x_0 x_n - x_0^2 x_{n-1}.$$

This establishes (7.2.7), from which it becomes clear that x_0 is even if and only if some x_n is even.

(iv) Analogously to (iii), one deduces the recurrence relation

$$y_{n+1} = 2x_0 y_n - y_{n-1}$$

with $y_{-1} \coloneqq 0$. So y_0 is odd if and only if some y_n is odd. □

7.2.7. Proposition. — *Let (x_1, y_1) and (x_2, y_2) be two solutions of the Pell equation $x^2 - Ny^2 = 1$ which satisfy $x_1 y_2 + x_2 y_1 = 0$, then $x_1 = \pm x_2$ and $y_1 = \mp y_2$.*

Proof. — By Brahmagupta's identity, we have

$$(x_1 x_2 + N y_1 y_2)^2 - N(x_1 y_2 + x_2 y_1)^2 = 1.$$

So $x_1 y_2 + x_2 y_1 = 0$ implies that $x_1 x_2 + N y_1 y_2 = \pm 1$. Multiplying this by y_2 yields $x_1 x_2 y_2 + N y_1 y_2^2 = \pm y_2$, and substituting the equation $x_1 y_2 = -x_2 y_1$ from the hypothesis yields

$$\pm y_2 = -x_2^2 y_1 + N y_1 y_2^2 = -y_1(x_2^2 - N y_2^2) = -y_2.$$

So the hypothesis $x_1 y_2 + x_2 y_1 = 0$ becomes $x_1 y_2 \mp x_2 y_2 = 0$, implying $x_1 = \pm x_2$ (which is also true in the case of the trivial solutions $(\pm 1, 0)$). □

7.2.8. Theorem. — *Assume that* $\mathrm{End}(A) = \mathbb{Z}$*, and let* $\deg(\lambda) = d^2$ *be the the degree of the polarization* $\lambda\colon A \to A^\vee$ *of minimal degree. Assume that either*

(i) 3 *divides* d*, and that* $d/3$ *is a perfect square, or*

(ii) $\gcd(3, d) = 1$ *and* $d \neq 1$*, and that the Pell equation* $x^2 - 3dy^2 = 1$ *has some solution with odd* y*.*

Then the generalized Kummer fourfolds $\mathrm{Kum}^2(A)$ *and* $\mathrm{Kum}^2(A^\vee)$ *are* not *birationally equivalent.*

Proof. — Assume for sake of contradiction that there exists some birational map $f\colon \mathrm{Kum}^2(A) \overset{\sim}{\dashrightarrow} \mathrm{Kum}^2(A^\vee)$. Now the general setup is exactly as in the proof of Namikawa's theorem (Theorem 7.2.1), whose notation we continue to use. That is, f induces via pullback on singular cohomology a lattice isometry

$$\varphi\colon \mathbb{Z}\check{h} \oplus \mathbb{Z}\check{\delta} \to \mathbb{Z}h \oplus \mathbb{Z}\delta,$$

where

$$\begin{cases} h^2 = 2d & (7.2.8)\\ \delta^2 = -6 & (7.2.9)\\ h.\delta = 0, & (7.2.10)\end{cases}$$

and similarly for $\check{h}$ and $\check{\delta}$. The goal is to show that $\varphi = \mathrm{id}$, and conclude as in Theorem 7.2.1 the non-birationality by arriving at a contradiction. We write

$$\begin{cases} \varphi(\check{h}) = xh + y\delta & (7.2.11)\\ \varphi(\check{\delta}) = zh + w\delta & (7.2.12)\end{cases}$$

for some $x, y, z, w \in \mathbb{Z}$. Since φ is an isometry we obtain from (7.2.8)-(7.2.10) the following equations:

$$\begin{cases} -6 = (zh + w\delta)^2 = 2dz^2 - 6w^2 & (7.2.13)\\ 2d = (xh + y\delta)^2 = 2dx^2 - 6y^2 & (7.2.14)\\ 0 = (xh + y\delta)(zh + w\delta) = 2dxz - 6yw & (7.2.15)\end{cases}$$

(i) We can write $d = 3d'$, since 3 divides d by assumption. So the system (7.2.13)-(7.2.15) becomes, after substituting d, dividing by 6, and slight rearranging,

$$\begin{cases} w^2 - d'z^2 = 1 & (7.2.16)\\ y^2 - d'(x^2 - 1) = 0 & (7.2.17)\\ yw - d'xz = 0 & (7.2.18)\end{cases}$$

Now we are already done, since the Pell equation $w^2 - d'z^2 = 1$ has only the trivial solutions $(\pm 1, 0)$ when d' is a perfect square.

We still push the calculation a bit further for the convenience of the reader who wishes to study this case even further. Equation (7.2.18) times w implies $yw^2 - d'xzw = 0$, and substituting (7.2.16) into it yields $y(d'z^2 + 1) - d'xzw = 0$. So d' divides y, say $y = d'y'$ for some $y' \in \mathbb{Z}$. Then (7.2.17) becomes $d'^2y'^2 - d'(x^2 - 1) = 0$, so $d'y'^2 - x^2 + 1 = 0$

since $d' \neq 0$. Also (7.2.18) becomes $d'y'w - d'xz = 0$, so $y'w - xz = 0$. Collecting this together, the system (7.2.16)-(7.2.18) is equivalent to

$$\begin{cases} w^2 - d'z^2 = 1 & (7.2.19) \\ x^2 - d'y'^2 = 1 & (7.2.20) \\ y'w - xz = 0 & (7.2.21) \\ y = d'y'. & (7.2.22) \end{cases}$$

Using Proposition 7.2.7 we conclude that

$$\boxed{\begin{cases} \varphi(\check{h}) = xh + d'y'\delta & (7.2.23) \\ \varphi(\check{\delta}) = \pm(y'h + x\delta) & (7.2.24) \end{cases}}$$

where (x, y') is a solution of the Pell equation $x^2 - d'y'^2 = 1$. But when d' is a perfect square, then the latter Pell equation has only the trivial solutions $(\pm 1, 0)$, so $\varphi(\check{h}) = \pm h$ and $\varphi(\check{\delta}) = \pm\delta$, where the minus signs can be excluded since h, $\check{h}$, δ, and $\check{\delta}$ are represented by effective divisors on projective varieties.

(ii) From now on we assume that $\gcd(3, d) = 1$. The system (7.2.13)-(7.2.15) becomes

$$\begin{cases} dz^2 - 3(w^2 - 1) = 0 & (7.2.25) \\ d(x^2 - 1) - 3y^2 = 0 & (7.2.26) \\ dxz - 3yw = 0. & (7.2.27) \end{cases}$$

By (7.2.25) we know that $3|dz^2$, so 3 divides z, say $z = 3z'$. By (7.2.27) we know that $d|yw$, and by (7.2.25) we have $d|w^2 - 1$, so $\gcd(d, w) = 1$ and d divides y, say $y = dy'$. With these changes of variables, the system (7.2.25)-(7.2.27) becomes

$$\begin{cases} 9dz'^2 - 3(w^2 - 1) = 0 & (7.2.28) \\ d(x^2 - 1) - 3d^2y'^2 = 0 & (7.2.29) \\ 3dxz' - 3dy'w = 0, & (7.2.30) \end{cases}$$

and after dividing by 3 and d respectively we get

$$\begin{cases} w^2 - 3dz'^2 = 1 & (7.2.31) \\ x^2 - 3dy'^2 = 1 & (7.2.32) \\ xz' - y'w = 0. & (7.2.33) \end{cases}$$

Taking Proposition 7.2.7 into account, we arrive at

$$\boxed{\begin{cases} \varphi(\check{h}) = xh + y'd\delta & (7.2.34) \\ \varphi(\check{\delta}) = \pm(3y'h + x\delta), & (7.2.35) \end{cases}}$$

where (x, y') is a solution of the Pell equation $x^2 - 3dy'^2 = 1$.

Now we will extract further conditions on the pairs (x, y') from the fact that φ is induced by a birational map. We follow the strategy of [Oka21, Prop. 2.2], where the non-birationality of Hilbert schemes of points on K3 surfaces is studied. Since $\mathrm{NS}(A) \simeq \mathbb{Z}$ and $\gcd(3, d) = 1$, we know by [Mor21, Thm. 0.1] that the moveable cone of $\mathrm{Kum}^2(A)$ is

$$\overline{\mathrm{Mov}}(\mathrm{Kum}^2(A)) = \mathbb{R}_{\geq 0} \cdot h \oplus \mathbb{R}_{\geq 0} \cdot \left(h - \frac{dy_0'}{3x_0}\delta \right),$$

where (x_0, y_0') is the fundamental solution of the diophantine equation $3x_0^2 - dy_0'^2 = 3$, and similarly for $\mathrm{Kum}^2(A^\vee)$. The latter equation implies that y_0' is divisible by 3, say $y_0' = 3y_0$. So

$$\overline{\mathrm{Mov}}(\mathrm{Kum}^2(A)) = \mathbb{R}_{\geq 0} \cdot h \oplus \mathbb{R}_{\geq 0} \cdot (x_0 h - dy_0 \delta),$$

where (x_0, y_0) is the fundamental solution of the Pell equation $x_0^2 - 3dy_0^2 = 1$. Since φ is induced from a birational equivalence of hyperkähler varieties,

$$\varphi \otimes \mathbb{R} \colon \ \mathrm{NS}(\mathrm{Kum}^2(A^\vee)) \otimes \mathbb{R} \to \mathrm{NS}(\mathrm{Kum}^2(A)) \otimes \mathbb{R}$$

maps the moveable cone to the movable cone. Hence φ preserves the set $\{h, x_0 h - dy_0\delta\}$ of primitive generators of the extremal rays of the moveable cone. That is, either $\varphi(\check{h}) = h$ or $\varphi(\check{h}) = x_0 h - dy_0\delta$. In the first case, we get $\varphi(\check{\delta}) = \delta$, as $\varphi(\check{\delta}) = -\delta$ is geometrically impossible, as before. In the second case, comparing with (7.2.34) yields that $(x, y') = (x_0, -y_0)$, so

$$\begin{cases} \varphi(\check{h}) = x_0 h - y_0 d\delta & (7.2.36) \\ \varphi(\check{\delta}) = \pm(3y_0 h - x_0\delta). & (7.2.37) \end{cases}$$

We claim that the undetermined sign in (7.2.37) must be positive, so either $\varphi = \mathrm{id}$ or

$$\boxed{\begin{cases} \varphi(\check{h}) = x_0 h - y_0 d\delta \\ \varphi(\check{\delta}) = 3y_0 h - x_0\delta, \end{cases}} \quad \begin{matrix} (7.2.38) \\ (7.2.39) \end{matrix}$$

where (x_0, y_0) is the fundamental solution of the Pell equation $x^2 - 3dy^2 = 1$. Indeed, consider the birational map

$$f^{-1} \colon \ \mathrm{Kum}^2(A^\vee) \dashrightarrow \mathrm{Kum}^2(A),$$

which induces $\varphi^{-1} = (f^{-1})^*$. By the above, either $\varphi^{-1} = \mathrm{id}$ (and thus $\varphi = \mathrm{id}$) or both φ^{-1} and φ correspond to the fundamental solution (x_0, y_0), up to sign. There are three cases to inspect: (a) Both φ and φ^{-1} have positive sign, (b) φ has positive sign and φ^{-1} has negative sign, (c) both φ and φ^{-1} have negative sign. In case (b) and (c) we calculate

$$\begin{aligned} \varphi^{-1}(\varphi(\check{h})) = \varphi^{-1}(x_0 h - y_0 d\delta) &= x_0(x_0\check{h} - y_0 d\check{\delta}) + y_0 d(3y_0\check{h} - x_0\check{\delta}) \\ &= (x_0^2 + 3dy_0^2)\check{h} - 2x_0 y_0 d\check{\delta} \neq \check{h}. \end{aligned}$$

Case (a) remains possible, since $\varphi^{-1}(\varphi(\check{h})) = \check{h}$ and $\varphi^{-1}(\varphi(\check{\delta})) = \check{\delta}$ does indeed hold.

Next, we want to see that $\varphi = \mathrm{id}$ as soon as y_0 is odd. Since $\varphi = f^*$ is induced from a birational map $f\colon \mathrm{Kum}^2(A) \dashrightarrow \mathrm{Kum}^2(A^\vee)$, we have a commutative diagram

$$\begin{array}{ccccccccc} \mathbb{Z}\check{h} \oplus \mathbb{Z}\check{\delta} & \simeq & \mathrm{NS}(\mathrm{Kum}^2(A^\vee)) & \hookrightarrow & \mathrm{H}^2(\mathrm{Kum}^2(A^\vee),\mathbb{Z}) & \simeq & \mathrm{H}^2(A^\vee,\mathbb{Z}) \oplus \mathbb{Z}\delta & \simeq & \mathrm{U}^{\oplus 3} \oplus \langle -6\rangle \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & & & \downarrow{\scriptstyle \varphi} \\ \mathbb{Z}h \oplus \mathbb{Z}\delta & \simeq & \mathrm{NS}(\mathrm{Kum}^2(A)) & \hookrightarrow & \mathrm{H}^2(\mathrm{Kum}^2(A),\mathbb{Z}) & \simeq & \mathrm{H}^2(A,\mathbb{Z}) \oplus \mathbb{Z}\delta & \simeq & \mathrm{U}^{\oplus 3} \oplus \langle -6\rangle. \end{array} \tag{7.2.40}$$

That is, the lattice isometry $\varphi\colon \mathbb{Z}\check{h} \oplus \mathbb{Z}\check{\delta} \to \mathbb{Z}h \oplus \mathbb{Z}\delta$ is the restriction of a lattice isometry $\mathrm{U}^{\oplus 3} \oplus \langle -6\rangle \xrightarrow{\sim} \mathrm{U}^{\oplus 3} \oplus \langle -6\rangle$. Denote the generator of $\langle -6\rangle$ again by δ, and denote the bases of the three copies of U by $\{\mathrm{e},\mathrm{f}\}$, $\{\mathrm{e}',\mathrm{f}'\}$, and $\{\mathrm{e}'',\mathrm{f}''\}$ respectively. Recall that under the identifications in Diagram (7.2.40), we have

$$\begin{aligned} \mathbb{Z}h \oplus \mathbb{Z}\delta &\hookrightarrow \mathrm{U}^{\oplus 3} \oplus \langle -6\rangle \\ h &\mapsto \mathrm{e} + d\mathrm{f} \\ \delta &\mapsto \delta, \end{aligned}$$

and similarly for $\check{h}$ and $\check{\delta}$. For the readers convenience, we recall that $\mathrm{e}^2 = 0$, $\mathrm{f}^2 = 0$ and $\mathrm{e}.\mathrm{f} = 1$, and that all directs sums in (7.2.40) are orthogonal direct sums.

Now, let (x_0, y_0) be (the fundamental) solution of $x^2 - 3dy^2 = 1$, and assume $\varphi(\check{\delta}) = 3y_0 h - x_0\delta$ and $\varphi(\mathrm{e} + d\mathrm{f}) = x_0(\mathrm{e} + d\mathrm{f}) - y_0 d\delta$. We study when it is possible to extend φ to a map of lattices $\mathrm{U} \oplus \langle -6\rangle \to \mathrm{U}^{\oplus 3} \oplus \langle -6\rangle$. Write

$$\varphi(\mathrm{f}) = x\mathrm{f} + yh + z\delta \quad + \quad v'\mathrm{e}' + w'\mathrm{f}' + v''\mathrm{e}'' + w''\mathrm{f}''$$

for some $x, y, z, v', w', v'', w'' \in \mathbb{Z}$. One verifies $\varphi(\mathrm{f})^2 = 2xy + 2dy^2 - 6z^2 + 2v'w' + 2v''w''$, so $f^2 = 0$ induces the equation

$$xy + dy^2 - 3z^2 + w = 0 \quad \text{with} \quad w := v'w' + v''w''.$$

We have $\mathrm{f}.\delta = 0$, so $\varphi(\mathrm{f}).\varphi(\delta) = 3y_0 x + 6dy_0 y + 6x_0 z$ provides the equation

$$y_0 x + 2dy_0 y + 2x_0 z = 0.$$

From $\mathrm{f}.h = 1$ and $\varphi(\mathrm{f}).\varphi(h) = x_0 x + 2dx_0 y + 6dy_0 z$ we get

$$x_0 x + 2dx_0 y + 6dy_0 z = 1.$$

Thus we have the system of equations

$$\begin{cases} xy + dy^2 - 3z^2 + w = 0 & (7.2.41) \\ y_0 x + 2dy_0 y + 2x_0 z = 0 & (7.2.42) \\ x_0 x + 2dx_0 y + 6dy_0 z = 1 & (7.2.43) \end{cases}$$

Equation (7.2.42) tells us that $x = \frac{-2}{y_0}(dy_0 y + x_0 z)$, and substituting this into (7.2.43) yields

$$1 = \frac{-2x_0}{y_0}(dy_0 y + x_0 z) + 2dx_0 y + 6dy_0 z = \left(\frac{-2x_0^2}{y_0} + 6dy_0\right) z.$$

Since $y_0 \neq 0$, the latter is equivalent to $y_0 = (-2x_0^2+6dy_0^2) = (-2x_0^2+2x_0^2-2)z = -2z$, where we have used the Pell equation $3dy_0^2 = x_0^2 - 1$. So we conclude $z = -\frac{y_0}{2}$ and see that y_0 has to be an even number, which is clearly in contradiction with the hypothesis that y_0 is odd. □

7.2.9. Remark. — We want to remark that at the end of the proof of Theorem 7.2.8, when y_0 is allowed to be even, there is nothing in the way to extend φ to all of $\mathrm{U}^{\oplus 3} \oplus \langle -6\rangle$. Indeed, continuing to calculate we obtain $x = -2dy + x_0$, and substituting into (7.2.41), we get the equation $-dy^2 + x_0y - \frac{3}{4}y_0^2 + w = 0$, which has a solution by the sheer fact that we can take w to be any integer. Thus we can extend φ to an isometric embedding

$$\mathrm{U} \oplus \langle -6\rangle \hookrightarrow \mathrm{U}^{\oplus 3} \oplus \langle -6\rangle.$$

Going further, we allude to general lattice theory, as recalled in [HuyK3, Ch. 14]. The fact that one has $\mathrm{U} \oplus \mathrm{U}^{\perp} = \Lambda$ for any embedding of lattices $\mathrm{U} \hookrightarrow \Lambda$, cf. [HuyK3, Ex. 14.0.3], allows one to check that

$$(\mathrm{U} \oplus \langle -6\rangle) \oplus (\mathrm{U} \oplus \langle -6\rangle)^{\perp} = \mathrm{U}^{\oplus 3} \oplus \langle -6\rangle.$$

The last check is not completely formal and once again uses that y_0 is even. We conclude that

$$(\mathrm{U} \oplus \langle -6\rangle)^{\perp} \simeq \mathrm{U}^{\oplus 2},$$

since even, unimodular lattices of sign (n_+, n_-) with $n_{\pm} > 0$ are unique, cf. [HuyK3, Thm. 14.1.1]. In view of this, extending φ becomes trivial.

7.2.10. Remark. — Theorem 7.2.8 remains true when replacing $A^{\vee}$ by any abelian surface $\check{A}$ with $\mathrm{End}(\check{A}) = \mathbb{Z}$ and $A \not\simeq \check{A}$. Indeed, write $\mathrm{NS}(\check{A}) = \mathbb{Z} \cdot \check{h}$ with $\check{h}^2 = 2\check{d}$. Then the existence of a lattice isometry $\varphi \colon \mathbb{Z}\check{h} \oplus \mathbb{Z}\check{\delta} \to \mathbb{Z}h \oplus \mathbb{Z}\delta$, as considered in the proof of Theorem 7.2.8, implies the equality

$$-12d = \mathrm{disc}(\mathbb{Z}h \oplus \mathbb{Z}\delta) = \mathrm{disc}(\mathbb{Z}\check{h} \oplus \mathbb{Z}\check{\delta}) = -12\check{d}$$

of discriminants. So $d = \check{d}$, and the calculations in the proof of Theorem 7.2.8 are valid without any modification.

7.2.11. Proposition. — *Let A and $\check{A}$ be abelian surfaces, and let $n \geq 2$. If $\mathrm{Kum}^n(A)$ and $\mathrm{Kum}^n(\check{A})$ are birationally equivalent, then A and $\check{A}$ are derived equivalent.*

Proof. — This is a direct adaptation of [Plo07, Prop. 10] from the case of Hilbert schemes of points on K3 surfaces to the case of generalized Kummer varieties.

We know that a birational equivalence $\mathrm{Kum}^n(A) \overset{\sim}{\dashrightarrow} \mathrm{Kum}^n(\check{A})$ induces a Hodge isometry $\mathrm{H}^2(\mathrm{Kum}^n(\check{A}), \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(\mathrm{Kum}^n(A), \mathbb{Z})$. Since we have a Hodge isometry $\mathrm{H}^2(\mathrm{Kum}^n(A), \mathbb{Z}) \simeq \mathrm{H}^2(A, \mathbb{Z}) \oplus \mathbb{Z} \cdot \delta$ and δ is algebraic, this means that we obtain isometries of transcendental lattices

$$\mathrm{T}(A) \simeq \mathrm{T}(\mathrm{Kum}^n(A)) \simeq \mathrm{T}(\mathrm{Kum}^n(\check{A})) \simeq \mathrm{T}(\check{A}).$$

This implies that we have a derived equivalence $\mathbf{D}^{\mathrm{b}}(A) \simeq \mathbf{D}^{\mathrm{b}}(\check{A})$, by [BM01, Thm. 5.1]. □

7.2.12. Theorem. — *Assume that* $\mathrm{End}(A) = \mathbb{Z}$*, and let* $\deg(\lambda) = d^2$ *be the the degree of the polarization* $\lambda\colon A \to A^\vee$ *of minimal degree. Assume that either*

(i) 3 *divides* d*, and that* $d/3$ *is a perfect square, or*

(ii) $\gcd(3, d) = 1$ *and* $d \neq 1$*, and that the Pell equation* $x^2 - 3dy^2 = 1$ *has some solution with odd* y*.*

Then we have isomorphisms

$$\mathrm{Bir}(\mathrm{Kum}^2(A)) \simeq \mathrm{Aut}(\mathrm{Kum}^2(A)) \simeq A[3] \rtimes \mathrm{Aut}_{\mathrm{AV}}(A),$$

where $\mathrm{Bir}(\mathrm{Kum}^2(A))$ *is the group of birational autoequivalences, and the action of* $\mathrm{Aut}_{\mathrm{AV}}(A)$ *on* $A[3]$ *is the obvious one.*

Proof. — Let $n \geq 3$. In general, following [BNS11, §3.1], we have an injective homomorphism

$$A \rtimes \mathrm{Aut}_{\mathrm{AV}}(A) \simeq \mathrm{Aut}(A) \hookrightarrow \mathrm{Aut}(\mathrm{Hilb}^n(A)),$$

and one calls automorphisms in the image of this map *natural*. A natural automorphism of $\mathrm{Hilb}^n(A)$ restricts to an automorphism of $\mathrm{Kum}^{n-1}(A)$ if and only if it corresponds to an element in $A[3] \rtimes \mathrm{Aut}_{\mathrm{AV}}(A)$; these are called again *natural*. By [BNS11, Thm. 3, Cor. 5], this leads to an injective homomorphism

$$A[3] \rtimes \mathrm{Aut}_{\mathrm{AV}}(A) \hookrightarrow \mathrm{Aut}(\mathrm{Kum}^{n-1}(A))$$

whose image consists of those automorphisms $f\colon \mathrm{Kum}^{n-1}(A) \xrightarrow{\sim} \mathrm{Kum}^{n-1}(A)$ which satisfy $f(E) = E$. Recall here that $E \subset \mathrm{Kum}^{n-1}(A)$ denotes the exceptional divisor of the Hilbert–Chow morphism, cf. ¶7.2.3; there exists a class $\delta \in \mathrm{NS}(\mathrm{Kum}^{n-1}(A))$ satisfying $2\delta = [E]$, cf. ¶7.2.2. Now the condition $f(E) = E$ is equivalent to $f^*(\delta) = \delta$, since E is rigid, cf. ¶7.2.3.

Consider a birational equivalence $f\colon \mathrm{Kum}^2(A) \overset{\sim}{\dashrightarrow} \mathrm{Kum}^2(A)$. Exactly as in the proofs of Theorems 7.2.1 and 7.2.8, we see that

$$f^*\colon \mathrm{NS}(\mathrm{Kum}^2(A)) \xrightarrow{\sim} \mathrm{NS}(\mathrm{Kum}^2(A))$$

is the identity map. In particular, f^* fixes some ample class (e.g. $k \cdot h - \delta$ for $k \gg 1$), and since f is already an isomorphism in codimension 1, cf. ¶7.2.4, f extends to an automorphism of $\mathrm{Kum}^2(A)$. We conclude that

$$\mathrm{Bir}(\mathrm{Kum}^2(A)) \simeq \mathrm{Aut}(\mathrm{Kum}^2(A)).$$

Finally, since $f^*(\delta) = \delta$, the automorphism f must be natural. $\square$

APPENDIX A

Code listings

In the proof of Proposition 5.2.7 we encountered a concrete map $d\colon \mathrm{S}_4 \to \Gamma_4 \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ which we claimed to be a cocycle. This can be computed by hand, but is a bit tedious. So we implemented code in the GAP computer algebra system [GAP] which verifies that d is a cocycle via a brute force computation, see Listing A.1 and an excerpt of its output in Listing A.2.

Listing A.1. Verification of the cocycle appearing in the proof of Proposition 5.2.7

```
# Normalizes a representative, given as a 4-tupel, of an element in the dual
# standard representation Γ4 ⊗Z Z/4Z.
normalize_elem := function (v)
  return [(v[1]-v[4]) mod 4, (v[2]-v[4]) mod 4, (v[3]-v[4]) mod 4, 0];
end;

# Check the cocycle condition for a map d: S4 → Γ4 ⊗Z Z/4Z.
check_derivation_rule := function (d)
  local G, Elts, n, i, j, s, t, d_a, d_b;
  G := SymmetricGroup(4);
  Elts := Elements(G);
  n := Length(Elts);
  for i in [1..n] do
    for j in [1..n] do
      s := Elts[i];
      t := Elts[j];
      Print("Checking ", s, ", ", t, "\n");

      # Multiplication for cycles in GAP is backwards!
      # Also, GAP uses row vectors, hence a common matrix-vector product is v*M.
      d_a := normalize_elem(d(t*s));
      d_b := normalize_elem(d(s) + d(t)*PermutationMat(s,4));
      if d_a <> d_b then
        Print("Check failed\n");
        Print(d_a, "\n", d_b);
        return;
```

```
      fi;
    od;
  od;
  Print("Check complete");
end;

# The following defines a 1-cocylce from S_4 to the dual standard representation.
# It was extended from d((1,2))=[0,0,2,0], d((2,3))=[2,0,0,0], d((3,4))=[0,2,0,0].
d := function (s)
  if   s = ()          then return [0,0,0,0];
  elif s = (1,2)       then return [0,0,2,0];
  elif s = (2,3)       then return [2,0,0,0];
  elif s = (1,3)       then return [2,2,2,0];
  elif s = (1,2,3)     then return [0,2,2,0];
  elif s = (1,3,2)     then return [2,2,0,0];
  elif s = (3,4)       then return [0,2,0,0];
  elif s = (1,2)(3,4)  then return [2,0,2,0];
  elif s = (3,4,2)     then return [2,0,2,0];
  elif s = (4,1,3)     then return [2,0,2,0];
  elif s = (3,4,1,2)   then return [0,2,0,0];
  elif s = (3,4,2,1)   then return [0,2,0,0];
  elif s = (2,4)       then return [0,0,2,0];
  elif s = (4,1,2)     then return [0,0,0,0];
  elif s = (4,3,2)     then return [2,2,0,0];
  elif s = (1,3)(2,4)  then return [0,2,2,0];
  elif s = (4,3,1,2)   then return [2,2,2,0];
  elif s = (4,1,3,2)   then return [2,0,0,0];
  elif s = (1,4)       then return [0,0,2,0];
  elif s = (4,2,1)     then return [0,0,0,0];
  elif s = (4,1)(2,3)  then return [2,2,0,0];
  elif s = (4,3,1)     then return [0,2,2,0];
  elif s = (4,2,3,1)   then return [2,2,2,0];
  elif s = (4,3,2,1)   then return [2,0,0,0];
  fi;
end;
```

Listing A.2. Output of Listing A.1

```
# GAP 4.12.2 of 2022-12-18
gap> check_derivation_rule(d);
Checking (), ()
Checking (), (3,4)
Checking (), (2,3)
 ...
Checking (1,4)(2,3), (1,4)
Checking (1,4)(2,3), (1,4,2,3)
Checking (1,4)(2,3), (1,4)(2,3)
Check complete
```

In Listing A.3 we implement the standard representation Γ_n of S_n and its dual $\Gamma_n^\vee$ in GAP and compute a few cohomology groups. We have included this code in order to enable the reader to experiment concretely with the cohomology groups $\mathrm{H}^\bullet(\mathrm{S}_n, \Gamma_n)$ and $\mathrm{H}^\bullet(\mathrm{S}_n, \Gamma_n^\vee)$. The code uses the GAP package `hap`, cf. [GAPhap].

Listing A.3. Implementation of the (dual) standard representation in GAP

```
# GAP 4.12.2 of 2022-12-18
LoadPackage("hap");

# Compute cohomology up to degree d of S_n, for n > 2, with values in a representation
# S_n → GL(k,ℤ), which is specified on the generators (1,2,...,n) and (1,2).
SnRepCohomology := function (MakeRepGenerators,n,d)
  local Sn, SnReso, SnGens, RepGroupGens, RepGroup, Rep, Cplx;
  Sn := SymmetricGroup(n);
  SnReso := ResolutionFiniteGroup(Sn,d+1);
  SnGens := [CycleFromList([1..n]), (1,2)];
  RepGroupGens := MakeRepGenerators(n);
  RepGroup := Group(RepGroupGens);
  Rep := GroupHomomorphismByImages(Sn,RepGroup,SnGens,RepGroupGens);
  Cplx := HomToIntegralModule(SnReso,Rep);
  return List([0..d], i -> Cohomology(Cplx,i));
end;

# Compute the permutation matrices of the standard (n − 1)-dimensional representation
# of S_n corresponding to the generators (1,2,...,n) and (1,2).
MakeStdRepGroupGens := function (n)
  local gen1, gen2, i;
  gen2 := PermutationMat((1,2),n-1);
  gen1 := PermutationMat(CycleFromList([1..n-1]),n-1);
  for i in [1..n-1] do
    gen1[i][1] := -1;
  od;
  gen1 := TransposedMat(gen1);
  return [gen1,gen2];
end;

# Compute cohomology up to degree d of S_n, for n > 2, with values in the standard
# (n − 1)-dimensional representation.
StdRepCohomology := function (n,d)
  return SnRepCohomology(MakeStdRepGroupGens,n,d);
end;

StdRepCohomology(6,2);
> [ [  ], [ 6 ], [ 2 ] ] # H^0(S_6,Γ_6) = 0, H^1(S_6,Γ_6) = ℤ/6ℤ, H^2(S_6,Γ_6) = ℤ/2ℤ
StdRepCohomology(5,2);
> [ [  ], [ 5 ], [  ] ]  # H^0(S_5,Γ_5) = 0, H^1(S_5,Γ_5) = ℤ/5ℤ, H^2(S_5,Γ_5) = 0

```

```
# Compute the permutation matrices of the dual standard (n-1)-dimensional
# representation of S_n corresponding to the generators (1,2,...,n) and (1,2).
MakeStdDualRepGroupGens := function (n)
  local gen1, gen2, i;
  gen2 := PermutationMat((1,2),n-1);
  gen1 := PermutationMat(CycleFromList([1..n-1]),n-1);
  for i in [1..n-1] do
    gen1[n-1][i] := -1;
  od;
  gen1 := TransposedMat(gen1);
  return [gen1,gen2];
end;

# Compute cohomology up to degree d of S_n, for n > 2, with values in the dual standard
# (n-1)-dimensional representation.
StdDualRepCohomology := function (n,d)
  return SnRepCohomology(MakeStdDualRepGroupGens,n,d);
end;

StdDualRepCohomology(4,2);
> [ [  ], [  ], [ 2 ] ] # H^0(S_4,Γ_4^∨) = 0, H^1(S_4,Γ_4^∨) = 0, H^2(S_4,Γ_4^∨) = Z/2Z
StdDualRepCohomology(5,2);
> [ [  ], [  ], [  ] ]  # H^0(S_5,Γ_5^∨) = 0, H^1(S_5,Γ_5^∨) = 0, H^2(S_5,Γ_5^∨) = 0
```

A.0.1. — Regarding the correctness of the code in Listing A.3. Let $n \geq 3$. Recall that the symmetric group S_n is generated by the permutation cycles $\tau := (1\,2)$ and $\sigma := (1\,2\ldots n)$. Denote the standard basis of $\mathbb{Z}^n$ by $(\mathrm{e}_i)_i$. For the standard representation $\Gamma_n \hookrightarrow \mathbb{Z}^n$ we pick the basis $\mathrm{e}_i - \mathrm{e}_n$, with $i = 1, \ldots n-1$. Then τ and σ correspond to the matrices

$$\tau \mathrel{\widehat{=}} \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \quad \text{and} \quad \sigma \mathrel{\widehat{=}} \begin{pmatrix} -1 & -1 & & \ldots & -1 \\ 1 & 0 & & & 0 \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}.$$

For the dual standard representation $\Gamma_n^\vee \twoheadleftarrow \mathbb{Z}^n$ we choose the basis $[\mathrm{e}_i]$, with $i = 1, \ldots n-1$. Then τ and σ correspond to the matrices

$$\tau \mathrel{\widehat{=}} \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \quad \text{and} \quad \sigma \mathrel{\widehat{=}} \begin{pmatrix} 0 & & & \ldots & -1 \\ 1 & 0 & & & \\ & 1 & 0 & & \vdots \\ & & \ddots & \ddots & -1 \\ & & & 1 & -1 \end{pmatrix}.$$

APPENDIX B

Summaries

Summary

Generalized Kummer varieties are one of the few known deformation families of hyperkähler varieties. The latter can be viewed as a generalization of K3 surfaces to higher dimensions, and have enjoyed much interest from algebraic geometers in the last decades. In this thesis we study generalized Kummer varieties from the non-commutative viewpoint of their associated derived categories, and we are particularly interested in derived equivalences between them. Derived equivalence is a weaker equivalence relation than the notion of isomorphism, and conjecturally it is also weaker than birational equivalence in the setting of hyperkähler varieties.

In one of our main theorems, we exhibit derived equivalent 'dual' generalized Kummer varieties and find among them examples which are not birationally equivalent; thus answering a question raised by Namikawa, and going beyond the case of Kummer K3 surfaces, which was already considered in the literature. Furthermore, we contribute certain (short) exact sequences which provide many derived autoequivalences of generalized Kummer varieties.

We build upon the derived McKay correspondence and the equivariant approach employed by Ploog in his study of derived equivalence of Hilbert schemes of points on K3 surfaces. A central technical ingredient in this thesis is the focused study of the (dual) standard representation of the symmetric group from the viewpoint of integral representation theory and group cohomology. We explain how to compute the group cohomology in Nakaoka's stable range of these representations with arbitrary coefficients, given the group cohomology of the symmetric group itself.

Samenvatting

Gegeneraliseerde Kummer-variëteiten en deformaties daarvan zijn een van de weinige bekende typen hyperkähler-variëteiten. Deze kunnen gezien worden als een generalisatie van K3-oppervlakken naar hogere dimensies, en hebben de afgelopen decennia

veel belangstelling genoten van algebraïsch meetkundigen. In dit proefschrift bestuderen we gegeneraliseerde Kummer-variëteiten vanuit het niet-commutatieve perspectief van hun bijbehorende afgeleide categorieën, en we zijn vooral geïnteresseerd in afgeleide equivalenties tussen deze variëteiten. Afgeleide equivalentie is een zwakkere equivalentierelatie dan het begrip isomorfisme, en vermoedelijk is het ook zwakker dan birationale equivalentie in het geval van hyperkähler-variëteiten.

In een van onze belangrijkste stellingen tonen we afgeleide equivalente 'duale' gegeneraliseerde Kummer-variëteiten en vinden we voorbeelden die niet birationaal equivalent zijn; hiermee beantwoorden we een vraag van Namikawa en gaan we verder dan het geval van Kummer K3-oppervlakken, dat al in de literatuur is behandeld. Verder construeren we bepaalde (korte) exacte rijen die veel afgeleide auto-equivalenties van gegeneraliseerde Kummer-variëteiten geven.

We bouwen voort op de afgeleide McKay-correspondentie en de equivariante benadering van Ploog in zijn studie van afgeleide equivalenties van Hilbert-schema's van punten op K3-oppervlakken. Een centraal technisch ingrediënt in dit proefschrift is de gerichte studie van de (duale) standaardrepresentatie van de symmetrische groep vanuit het perspectief van de integrale representatietheorie en groepscohomologie. We leggen uit hoe de groepscohomologie van deze representaties in Nakaoka's stabiele bereik met willekeurige coëfficiënten kan worden berekend, gegeven de groepscohomologie van de symmetrische groep zelf.

Research Data Management

This thesis research has been carried out under the institute research data management policy of IMAPP, Radboud University. No data has been produced in this project.

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Curriculum Vitae

Pablo Magni was born in Bonn, Germany, on January 31st 1995. After obtaining his Abitur from the Amos-Comenius-Gymnasium in 2013, he enrolled at the University of Bonn, pursuing the study of mathematics. He obtained his Bachelor's Degree in Mathematics in 2016 and his Master's Degree in Mathematics in 2018, under the supervision of Daniel Huybrechts.

In November 2018, he moved to the Netherlands in order to work as a PhD candidate at Radboud University Nijmegen as part of the NWO project "Arithmetic and Geometry Beyond Shimura Varieties" led by Ben Moonen and Lenny Taelman. Starting from Spring 2020, he was supervised by Lenny Taelman and Lie Fu, after Ben Moonen became absent due to personal circumstances. This thesis is the product of research carried out in this later period.

Since June 2023, he is a postdoc in the research group of Charles Vial at the University of Bielefeld.

Bibliography

[Add16] N. Addington. "New derived symmetries of some hyperkähler varieties". *Algebr. Geom.* 3.2 (2016), pp. 223–260. DOI: 10.14231/AG-2016-011.

[ADM16] N. Addington, W. Donovan, and C. Meachan. "Moduli spaces of torsion sheaves on $K3$ surfaces and derived equivalences". *J. Lond. Math. Soc., II. Ser.* 93.3 (2016), pp. 846–865. DOI: 10.1112/jlms/jdw022.

[AM] A. Adem and R. J. Milgram. *Cohomology of finite groups.* 2nd ed. Vol. 309. Grundlehren Math. Wiss. Springer, Berlin, 2004.

[Ami18] Z. Amir-Khosravi. "Serre's tensor construction and moduli of abelian schemes". *Manuscr. Math.* 156.3-4 (2018), pp. 409–456. DOI: 10.1007/s00229-017-0976-x.

[AT14] N. Addington and R. Thomas. "Hodge theory and derived categories of cubic fourfolds". *Duke Math. J.* 163.10 (2014), pp. 1885–1927. DOI: 10.1215/00127094-2738639.

[Bal02] P. Balmer. "Presheaves of triangulated categories and reconstruction of schemes". *Math. Ann.* 324.3 (2002), pp. 557–580. DOI: 10.1007/s00208-002-0353-1.

[Bal09] M. R. Ballard. *Equivalences of derived categories of sheaves on quasi-projective schemes.* Version 2 (Sep 2009). 2009. arXiv: 0905.3148.

[Bal11] M. R. Ballard. "Derived categories of sheaves on singular schemes with an application to reconstruction". *Adv. Math.* 227.2 (2011), pp. 895–919. DOI: 10.1016/j.aim.2011.02.016.

[Bat99] V. V. Batyrev. "Birational Calabi-Yau n-folds have equal Betti numbers". In: *New trends in algebraic geometry. Selected papers presented at the Euro conference, Warwick, UK, July 1996.* Cambridge University Press, Cambridge, 1999, pp. 1–11.

[BB] A. Björner and F. Brenti. *Combinatorics of Coxeter Groups.* Vol. 231. Grad. Texts Math. Springer, New York, 2005. DOI: 10.1007/3-540-27596-7.

[BB03] A. Bondal and M. van den Bergh. “Generators and representability of functors in commutative and noncommutative geometry”. *Mosc. Math. J.* 3.1 (2003), pp. 1–36.

[BBH] C. Bartocci, U. Bruzzo, and D. Hernández Ruipérez. *Fourier-Mukai and Nahm Transforms in Geometry and Mathematical Physics*. Vol. 276. Prog. Math. Birkhäuser, Basel, 2009. DOI: 10.1007/b11801.

[Bea83] A. Beauville. “Variétés kähleriennes dont la première classe de Chern est nulle”. *J. Differ. Geom.* 18.4 (1983), pp. 755–782. DOI: 10.4310/jdg/1214438181.

[BeaCAS] A. Beauville. *Complex Algebraic Surfaces*. 2nd. Vol. 34. Lond. Math. Soc. Stud. Texts. Cambridge Univ. Press, Cambridge, 1996.

[Bec23] T. Beckmann. “Derived categories of hyper-Kähler manifolds: extended Mukai vector and integral structure”. *Compos. Math.* 159.1 (2023), pp. 109–152. DOI: 10.1112/S0010437X22007849.

[BerIGH] G. Berhuy. *An Introduction to Galois Cohomology and its Applications*. Vol. 377. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2010. DOI: 10.1017/CBO9781139107051.

[Bey82] F. R. Beyl. “Isoclinisms of group extensions and the Schur multiplicator”. In: *Groups—St. Andrews 1981*. Vol. 71. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1982, pp. 169–185.

[Bin21] W. Bindt. “Hyperkähler varieties and their relation to Shimura stacks”. PhD thesis. University of Amsterdam, 2021. URL: https://hdl.handle.net/11245.1/68377e9e-042c-4d1b-9fd2-42f437f7153f.

[BKR01] T. Bridgeland, A. King, and M. Reid. “The McKay correspondence as an equivalence of derived categories”. *J. Am. Math. Soc.* 14.3 (2001), pp. 535–554. DOI: 10.1090/S0894-0347-01-00368-X.

[BL] J. Bernstein and V. Lunts. *Equivariant Sheaves and Functors*. Vol. 1578. Lect. Notes Math. Springer, Berlin, 1994. DOI: 10.1007/BFb0073549.

[BL03] C. Birkenhake and H. Lange. “An isomorphism between moduli spaces of abelian varieties”. *Math. Nachr.* 253 (2003), pp. 3–7. DOI: 10.1002/mana.200310041.

[Blu07] M. Blume. “McKay correspondence and G-Hilbert schemes”. PhD thesis. Eberhard-Karls-Universität Tübingen, 2007. URL: https://d-nb.info/98519670X.

[Blu11] M. Blume. “Construction of G-Hilbert schemes”. *Math. Nachr.* 284.8-9 (2011), pp. 953–959. DOI: 10.1002/mana.200710234.

[BM01] T. Bridgeland and A. Maciocia. “Complex surfaces with equivalent derived categories”. *Math. Z.* 236.4 (2001), pp. 677–697. DOI: 10.1007/PL00004847.

[BNS11] S. Boissière, M. Nieper-Wißkirchen, and A. Sarti. “Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties”. *J. Math. Pures Appl. (9)* 95.5 (2011), pp. 553–563. DOI: 10.1016/j.matpur.2010.12.003.

[BO01] A. Bondal and D. Orlov. "Reconstruction of a variety from the derived category and groups of autoequivalences". *Compos. Math.* 125.3 (2001), pp. 327–344. DOI: 10.1023/A:1002470302976.

[BO95] A. Bondal and D. Orlov. *Semiorthogonal decomposition for algebraic varieties.* Version 1 (Jun 1995). 1995. arXiv: alg-geom/9506012.

[Bog74] F. A. Bogomolov. "The decomposition of Kähler manifolds with a trivial canonical class". *Mat. Sb., Nov. Ser.* 93.135 (1974), pp. 573–575.

[Bri02] T. Bridgeland. "Flops and derived categories." *Invent. Math.* 147.3 (2002), pp. 613–632. DOI: 10.1007/s002220100185.

[Bri77] J. Briancon. "Description de $\mathrm{Hilb}^n \mathrm{C}\{x, y\}$". *Invent. Math.* 41 (1977), pp. 45–89. DOI: 10.1007/BF01390164.

[BroCG] K. S. Brown. *Cohomology of Groups.* Vol. 87. Graduate Texts in Mathematics. Springer, New York, 1982.

[BS64] A. Borel and J.-P. Serre. "Théoremes de finitude en cohomologie galoisienne". *Comment. Math. Helv.* 39 (1964), pp. 111–164. DOI: 10.1007/BF02566948.

[Che15] X.-W. Chen. "A note on separable functors and monads with an application to equivariant derived categories". *Abh. Math. Semin. Univ. Hamb.* 85.1 (2015), pp. 43–52. DOI: 10.1007/s12188-015-0103-4.

[CHN10] F. R. Cohen, D. J. Hemmer, and D. K. Nakano. "On the cohomology of Young modules for the symmetric group." *Adv. Math.* 224.4 (2010), pp. 1419–1461. DOI: 10.1016/j.aim.2010.01.004.

[CohAlg] P. M. Cohn. *Algebra. Volume 1.* 2nd. John Wiley & Sons, Ltd., Chichester, 1982.

[Con04] B. Conrad. "Gross-Zagier revisited (with an appendix by W. R. Mann)". In: *Heegner points and Rankin L-series. Papers from the workshop on special values of Rankin L-series, Berkeley, CA, USA, December 2001.* Cambridge University Press, Cambridge, 2004, pp. 67–163.

[Del97] P. Deligne. "Action du groupe des tresses sur une catégorie". *Invent. Math.* 128.1 (1997), pp. 159–175. DOI: 10.1007/s002220050138.

[EGM] B. Edixhoven, G. van der Geer, and B. Moonen. *Abelian Varieties.* Preliminary version of the first chapters. URL: https://www.math.ru.nl/~bmoonen/research.html#bookabvar.

[Ela11] A. Elagin. "Cohomological descent theory for a morphism of stacks and for equivariant derived categories". *Sb. Math.* 202.4 (2011), pp. 495–526. DOI: 10.1070/SM2011v202n04ABEH004153.

[Ela14] A. Elagin. *On equivariant triangulated categories.* Version 2 (Oct 2015). 2014. arXiv: 1403.7027.

[FGAex] B. Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure, and A. Vistoli. *Fundamental algebraic geometry: Grothendieck's FGA explained.* Vol. 123. Math. Surv. Monogr. American Mathematical Society (AMS), Providence, RI, 2005.

[FH23] S. Frei and K. Honigs. "Groups of symplectic involutions on symplectic varieties of Kummer type and their fixed loci". *Forum Math. Sigma* 11 (2023). Id/No e40, p. 35. DOI: 10.1017/fms.2023.37.

[Fog68] J. Fogarty. "Algebraic families on an algebraic surface". *Am. J. Math.* 90 (1968), pp. 511–521. DOI: 10.2307/2373541.

[FuECT] L. Fu. *Étale Cohomology Theory.* 2nd revised ed. Vol. 14. Nankai Tracts Math. World Scientific, Hackensack, NJ, 2015. DOI: 10.1142/9569.

[Fuj87] A. Fujiki. "On the de Rham cohomology group of a compact Kähler symplectic manifold". In: *Algebraic geometry, Sendai, 1985.* Vol. 10. Adv. Stud. Pure Math. North-Holland, Amsterdam, 1987, pp. 105–165. DOI: 10.2969/aspm/01010105.

[Gab62] P. Gabriel. "Des catégories abéliennes". *Bull. Soc. Math. Fr.* 90 (1962), pp. 323–448. DOI: 10.24033/bsmf.1583.

[GAP] The GAP Group. *GAP – Groups, Algorithms, and Programming.* Version 4.12.0. 2022. URL: https://www.gap-system.org.

[GAPhap] G. Ellis. *HAP, Homological Algebra Programming.* Version 1.47. Refereed GAP package. Aug. 2022. URL: https://gap-packages.github.io/hap.

[GirCNA] J. Giraud. *Cohomologie Non Abélienne.* Vol. 179. Grundlehren der Mathematischen Wissenschaften. Springer, Berlin, 1971.

[Gro57] A. Grothendieck. "Sur quelques points d'algèbre homologique". *Tôhoku Math. J. (2)* 9 (1957), pp. 119–221.

[GS93] L. Göttsche and W. Soergel. "Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces". *Math. Ann.* 296.2 (1993), pp. 235–245. DOI: 10.1007/BF01445104.

[Hai01] M. Haiman. "Hilbert schemes, polygraphs and the Macdonald positivity conjecture". *J. Am. Math. Soc.* 14.4 (2001), pp. 941–1006. DOI: 10.1090/S0894-0347-01-00373-3.

[Hal22] J. Hall. *Further remarks on derived categories of algebraic stacks.* Version 1 (May 2022). 2022. arXiv: 2205.09312.

[HarAS] R. Hartshorne. *Ample subvarieties of algebraic varieties. Notes written in collaboration with C. Musili.* Vol. 156. Lect. Notes Math. Springer, Cham, 1970. DOI: 10.1007/BFb0067839.

[Hem09] D. J. Hemmer. "Cohomology and generic cohomology of Specht modules for the symmetric group." *J. Algebra* 322.5 (2009), pp. 1498–1515. DOI: 10.1016/j.jalgebra.2009.05.041.

[HL] D. Huybrechts and M. Lehn. *The Geometry of Moduli Spaces of Sheaves.* 2nd ed. Cambridge University Press, Cambridge, 2010.

[HLOY03] S. Hosono, B. H. Lian, K. Oguiso, and S.-T. Yau. “Kummer structures on a $K3$ surface: An old question of T. Shioda”. *Duke Math. J.* 120.3 (2003), pp. 635–647. DOI: 10.1215/S0012-7094-03-12036-0.

[HLT21] K. Honigs, M. Lieblich, and S. Tirabassi. “Fourier-Mukai partners of Enriques and bielliptic surfaces in positive characteristic”. *Math. Res. Lett.* 28.1 (2021), pp. 65–91. DOI: 10.4310/MRL.2021.v28.n1.a3.

[HN11] D. Huybrechts and M. Nieper-Wisskirchen. “Remarks on derived equivalences of Ricci-flat manifolds”. *Math. Z.* 267.3-4 (2011), pp. 939–963. DOI: 10.1007/s00209-009-0655-z.

[HR15] J. Hall and D. Rydh. “Algebraic groups and compact generation of their derived categories of representations”. *Indiana Univ. Math. J.* 64.6 (2015), pp. 1903–1923. DOI: 10.1512/iumj.2015.64.5719.

[HR74] M. Hochster and J. L. Roberts. “Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay”. *Adv. Math.* 13 (1974), pp. 115–175. DOI: 10.1016/0001-8708(74)90067-X.

[Huy99] D. Huybrechts. “Compact hyperkähler manifolds: Basic results”. *Invent. Math.* 135.1 (1999), pp. 63–113. DOI: 10.1007/s002220050280.

[HuyFM] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. Clarendon Press, Oxford University Press, Oxford, 2006. DOI: 10.1093/acprof:oso/9780199296866.001.0001.

[HuyK3] D. Huybrechts. *Lectures on K3 surfaces*. Vol. 158. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016. DOI: 10.1017/CBO9781316594193.

[Iar72] A. Iarrobino. “Punctual Hilbert schemes”. *Bull. Am. Math. Soc.* 78 (1972), pp. 819–823. DOI: 10.1090/S0002-9904-1972-13049-0.

[IN96] Y. Ito and I. Nakamura. “McKay correspondence and Hilbert schemes”. *Proc. Japan Acad., Ser. A* 72.7 (1996), pp. 135–138. DOI: 10.3792/pjaa.72.135.

[IshIS] S. Ishii. *Introduction to Singularities*. 2nd edition. Springer, Tokyo, 2018. DOI: 10.1007/978-4-431-56837-7.

[IveCS] B. Iversen. *Cohomology of Sheaves*. Universitext. Springer, Berlin, 1986. DOI: 10.1007/978-3-642-82783-9.

[JKPell] M. J. j. Jacobson and H. C. Williams. *Solving the Pell Equation*. CMS Books Math./Ouvrages Math. SMC. Springer, New York, 2009. DOI: 10.1007/978-0-387-84923-2.

[Kal09] D. Kaledin. *Non-commutative Geometry from a homological point of view*. KIAS, Seoul. Oct. 2009. URL: http://imperium.lenin.ru/~kaledin/seoul/.

[KarGR] G. Karpilovsky. *Group Representations. Volume 2*. Vol. 177. North-Holland Math. Stud. North-Holland, Amsterdam, 1993.

[KarSN] G. Karpilovsky. *The Schur Multiplier*. Vol. 2. London Mathematical Society Monographs. New Series. Clarendon Press, Oxford, 1987.

[Kaw02] Y. Kawamata. "D-equivalence and K-equivalence". *J. Differ. Geom.* 61.1 (2002), pp. 147–171. DOI: 10.4310/jdg/1090351323.

[Kaw04] Y. Kawamata. "Equivalences of derived categories of sheaves on smooth stacks". *Am. J. Math.* 126.5 (2004), pp. 1057–1083. DOI: 10.1353/ajm.2004.0036.

[Kaw18] Y. Kawamata. "Birational geometry and derived categories". In: *Celebrating the 50th anniversary of the Journal of Differential Geometry. Lectures given at the Geometry and Topology Conference at Harvard University, Cambridge, MA, USA, April 28 – May 2, 2017.* International Press, Somerville, MA, 2018, pp. 291–317. DOI: 10.4310/SDG.2017.v22.n1.a11.

[KM17] A. Krug and C. Meachan. *Universal functors on symmetric quotient stacks of Abelian varieties*. Version 2 (Aug 2019). 2017. arXiv: 1710.08618.

[KS09] M. Kontsevich and Y. Soibelman. "Notes on A_∞-algebras, A_∞-categories and non-commutative geometry". In: *Homological mirror symmetry. New developments and perspectives*. Springer, Berlin, 2009, pp. 153–219. DOI: 10.1007/978-3-540-68030-7_6.

[KS99] A. S. Kleshchev and J. Sheth. "On extensions of simple modules over symmetric and algebraic groups". *J. Algebra* 221.2 (1999), pp. 705–722. DOI: 10.1006/jabr.1999.8038.

[Kup21] A. Kupers. *Homological stability minicourse*. May 2021. URL: https://www.utsc.utoronto.ca/people/kupers/seminars/minicourse-on-homological-stability/.

[LanAV] S. Lang. *Abelian Varieties*. Reprint. Springer, New York-Berlin, 1983.

[LiuAG] Q. Liu. *Algebraic Geometry and Arithmetic Curves*. Vol. 6. Oxf. Grad. Texts Math. Oxford University Press, Oxford, 2006.

[LM] G. Laumon and L. Moret-Bailly. *Champs algébriques*. Vol. 39. Ergeb. Math. Grenzgeb., 3. Folge. Springer, Berlin, 2000.

[LO15] M. Lieblich and M. Olsson. "Fourier-Mukai partners of $K3$ surfaces in positive characteristic". *Ann. Sci. Éc. Norm. Supér. (4)* 48.5 (2015), pp. 1001–1033.

[LT17] A. C. López Martín and C. Tejero Prieto. "Derived equivalences of Abelian varieties and symplectic isomorphisms". *J. Geom. Phys.* 122 (2017), pp. 92–102. DOI: 10.1016/j.geomphys.2017.01.010.

[MacSF] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. 2nd. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1998.

[Mea15] C. Meachan. "Derived autoequivalences of generalised Kummer varieties". *Math. Res. Lett.* 22.4 (2015), pp. 1193–1221. DOI: 10.4310/MRL.2015.v22.n4.a12.

[MFG] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory.* 3rd enl. ed. Vol. 34. Ergeb. Math. Grenzgeb. Springer, Berlin, 1994. URL: https://hdl.handle.net/2433/102881.

[Mil86] J. S. Milne. “Abelian varieties”. In: *Arithmetic Geometry.* Ed. by G. Cornell and J. Silverman. Springer, New York, 1986, pp. 103–150.

[MMY20] C. Meachan, G. Mongardi, and K. Yoshioka. “Derived equivalent Hilbert schemes of points on $K3$ surfaces which are not birational”. *Math. Z.* 294.3-4 (2020), pp. 871–880. DOI: 10.1007/s00209-019-02281-1.

[MO] The MathOverflow Community. *MathOverflow.* URL: https://mathoverflow.net.

[Mor21] A. Mori. “Nef cone of a generalized Kummer 4-fold”. *Hokkaido Math. J.* 50.2 (2021), pp. 151–163. DOI: 10.14492/hokmj/2018-919.

[Mor77] S. Mori. “The endomorphism rings of some abelian varieties. II”. *Jpn. J. Math., New Ser.* 3 (1977), pp. 105–109.

[Mor84] D. R. Morrison. “On K3 surfaces with large Picard number”. *Invent. Math.* 75 (1984), pp. 105–121. DOI: 10.1007/BF01403093.

[Muk78] S. Mukai. “Semi-homogeneous vector bundles on an abelian variety”. *J. Math. Kyoto Univ.* 18.2 (1978), pp. 239–272. DOI: 10.1215/kjm/1250522574.

[Muk81] S. Mukai. “Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves”. *Nagoya Math. J.* 81 (1981), pp. 153–175. DOI: 10.1017/S002776300001922X.

[Muk87] S. Mukai. “On the moduli space of bundles on $K3$ surfaces. I”. In: *Vector bundles on algebraic varieties (Bombay, 1984).* Vol. 11. Tata Inst. Fund. Res. Stud. Math. Tata Inst. Fund. Res., Bombay, 1987, pp. 341–413.

[Muk98] S. Mukai. *Abelian variety and spin representation.* Warwick. 1998. URL: https://www.kurims.kyoto-u.ac.jp/~mukai/paper/warwick13.pdf.

[Mum66] D. Mumford. “On the equations defining Abelian varieties. I”. *Invent. Math.* 1 (1966), pp. 287–354. DOI: 10.1007/BF01389737.

[MumAV] D. Mumford. *Abelian Varieties. With appendices by C. P. Ramanujam and Yuri Manin.* Corrected reprint of the 2nd ed. 1974. Hindustan Book Agency, New Delhi/distrib. by American Mathematical Society (AMS); Tata Institute of Fundamental Research, Bombay, 2008.

[Nak60] M. Nakaoka. “Decomposition theorem for homology groups of symmetric groups”. *Ann. Math. (2)* 71 (1960), pp. 16–42. DOI: 10.2307/1969878.

[Nam02a] Y. Namikawa. “Counter-example to global Torelli problem for irreducible symplectic manifolds”. *Math. Ann.* 324.4 (2002), pp. 841–845. DOI: 10.1007/s00208-002-0344-2.

[Nam02b] Y. Namikawa. “Erratum: “Counter-example to global Torelli problem for irreducible symplectic manifolds””. *Math. Ann.* 324.4 (2002), p. 847. DOI: 10.1007/s00208-002-0344-2.

[Nik76] V. V. Nikulin. "On Kummer surfaces". *Math. USSR, Izv.* 9 (1976), pp. 261–275. DOI: 10.1070/IM1975v009n02ABEH001477.

[NSW] J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of number fields*. 2nd ed. Vol. 323. Grundlehren der Mathematischen Wissenschaften. Springer, Berlin, 2008. DOI: 10.1007/978-3-540-37889-1.

[OEIS] OEIS Foundation Inc. *The On-Line Encyclopedia of Integer Sequences*. Published electronically at http://oeis.org. 2023.

[OGr03] K. G. O'Grady. "A new six-dimensional irreducible symplectic variety". *J. Algebr. Geom.* 12.3 (2003), pp. 435–505. DOI: 10.1090/S1056-3911-03-00323-0.

[OGr97] K. G. O'Grady. "The weight-two Hodge structure of moduli spaces of sheaves on a K^3 surface". *J. Algebr. Geom.* 6.4 (1997), pp. 599–644.

[OGr99] K. G. O'Grady. "Desingularized moduli spaces of sheaves on a $K3$". *J. Reine Angew. Math.* 512 (1999), pp. 49–117. DOI: 10.1515/crll.1999.056.

[Oka21] S. Okawa. "An example of birationally inequivalent projective symplectic varieties which are D-equivalent and L-equivalent". *Math. Z.* 297.1-2 (2021), pp. 459–464. DOI: 10.1007/s00209-020-02519-3.

[OlsASS] M. Olsson. *Algebraic Spaces and Stacks*. Vol. 62. Colloq. Publ., Am. Math. Soc. American Mathematical Society (AMS), Providence, RI, 2016.

[OorCGS] F. Oort. *Commutative Group Schemes*. Vol. 15. Lect. Notes Math. Springer, Cham, 1966. DOI: 10.1007/BFb0097479.

[Orl02] D. O. Orlov. "Derived categories of coherent sheaves on abelian varieties and equivalences between them". *Izv. Math.* 66.3 (2002), pp. 569–594. DOI: 10.1070/IM2002v066n03ABEH000389.

[Orl97] D. O. Orlov. "Equivalences of derived categories and $K3$ surfaces". *J. Math. Sci., New York* 84.5 (1997), pp. 1361–1381. DOI: 10.1007/BF02399195.

[Plo05] D. Ploog. "Groups of autoequivalences of derived categories of smooth projective varieties". PhD thesis. Freie Universität Berlin, Institut für Mathematik und Informatik, 2005. URL: https://www.math.uni-bonn.de/people/huybrech/phd-ploog.ps.

[Plo07] D. Ploog. "Equivariant autoequivalences for finite group actions". *Adv. Math.* 216.1 (2007), pp. 62–74. DOI: 10.1016/j.aim.2007.05.002.

[Pol96] A. Polishchuk. "Symplectic biextensions and a generalization of the Fourier-Mukai transform". *Math. Res. Lett.* 3.6 (1996), pp. 813–828. DOI: 10.4310/MRL.1996.v3.n6.a9.

[PolAV] A. Polishchuk. *Abelian Varieties, Theta Functions, and the Fourier Transform*. Vol. 153. Camb. Tracts Math. Cambridge University Press, Cambridge, 2003.

[Qui74] D. Quillen. *Notebooks 1974-1*. 1974. URL: https://www.claymath.org/publications/quillen-notebooks.

[Rei97] M. Reid. *McKay correspondence*. Version 3 (Feb 1997). 1997. arXiv: alg-geom/9702016.

[RW17] O. Randal-Williams and N. Wahl. "Homological stability for automorphism groups". *Adv. Math.* 318 (2017), pp. 534–626. DOI: 10.1016/j.aim.2017.07.022.

[Sch11] I. Schur. "Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen." *J. Reine Angew. Math.* 139 (1911), pp. 155–250. DOI: 10.1515/crll.1911.139.155.

[Ser59] J.-P. Serre. "Morphismes universels et variété d'Albanese". *Séminaire Claude Chevalley* 4 (1958-1959), pp. 1–22.

[SerAGC] J.-P. Serre. *Algebraic Groups and Class Fields. Transl. of the French edition*. Vol. 117. Grad. Texts Math. Springer, New York, 1988.

[SerGC] J.-P. Serre. *Galois Cohomology*. 2nd printing. Springer Monographs in Mathematics. Translated from the French by Patrick Ion. Springer, Berlin, 2002.

[Shc04] V. V. Shchigolev. "On some extensions of completely splittable modules". *Izv. Math.* 68.4 (2004), pp. 833–850. DOI: 10.1070/IM2004v068n04ABEH000499.

[Shi78] T. Shioda. "The period map of Abelian surfaces". *J. Fac. Sci., Univ. Tokyo, Sect. I A* 25 (1978), pp. 47–59.

[Sos12] P. Sosna. "Linearisations of triangulated categories with respect to finite group actions". *Math. Res. Lett.* 19.5 (2012), pp. 1007–1020. DOI: 10.4310/MRL.2012.v19.n5.a4.

[SP] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu. 2022.

[Ste07] P. Stellari. "Derived categories and Kummer varieties". *Math. Z.* 256.2 (2007), pp. 425–441. DOI: 10.1007/s00209-006-0088-x.

[SW20] D. Sprehn and N. Wahl. "Homological stability for classical groups". *Trans. Am. Math. Soc.* 373.7 (2020), pp. 4807–4861. DOI: 10.1090/tran/8030.

[SzaGCF] T. Szamuely. *Galois Groups and Fundamental Groups*. Vol. 117. Camb. Stud. Adv. Math. Cambridge University Press, Cambridge, 2009.

[Tae19] L. Taelman. *Derived equivalences of hyperkähler varieties*. Version 5 (Feb 2021). 2019. arXiv: 1906.08081.

[Ver77] J.-L. Verdier. "Categories dérivées. Quelques résultats (Etat O)". In: *Semin. Geom. algebr. Bois-Marie, SGA 4 1/2*. Vol. 569. Lect. Notes Math. Springer, Berlin, 1977, pp. 262–311. DOI: 10.1007/BFb0091516.

[Ver96] J.-L. Verdier. *Des catégories dérivées des catégories abéliennes.* Vol. 239. Astérisque. Société Mathématique de France, Paris, 1996.

[Wah22] N. Wahl. *Homological stability: a tool for computations.* Version 1 (Mar 2022). 2022. arXiv: 2203.07767.

[WeiIHA] C. A. Weibel. *An Introduction to Homological Algebra.* Vol. 38. Camb. Stud. Adv. Math. Cambridge University Press, Cambridge, 1994.

[Yos01] K. Yoshioka. "Moduli spaces of stable sheaves on abelian surfaces." *Math. Ann.* 321.4 (2001), pp. 817–884. DOI: 10.1007/s002080100255.

[Yos16] K. Yoshioka. "Bridgeland's stability and the positive cone of the moduli spaces of stable objects on an abelian surface". In: *Development of moduli theory – Kyoto 2013. Proceedings of the 6th Mathematical Society of Japan-Seasonal Institute, MSJ-SI, Kyoto, Japan, June 11–21, 2013.* Mathematical Society of Japan (MSJ), Tokyo, 2016, pp. 473–537.

[Zar18] Y. G. Zarhin. "Endomorphism algebras of abelian varieties with special reference to superelliptic Jacobians". In: *Geometry, algebra, number theory, and their information technology applications.* Vol. 251. Springer Proc. Math. Stat. Springer, Cham, 2018, pp. 477–528. DOI: 10.1007/978-3-319-97379-1_22.